The Role of Endogenous Vintage Specific Depreciation on the Optimal Behavior of Firms*

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August 10, 2006

Abstract

This paper studies the firms' capital accumulation process in a vintage capital model with embodied technological change. We take into account that depreciation is endogenous and in particular associated with vintage specific maintenance expenditure. We prove that maintenance is a local substitute for investment as soon as the marginal cost of maintenance is strictly increasing. We show that maintenance and investment in new capital goods appear as complements with respect to the changes in some exogenous factors such as productivity, cost of maintenance, fixed cost of operation, efficiency of maintenance services and appear as substitutes with respect to the price of new machines. Allowing for investment in old vintages, we establish that investment in old machines appears as a substitute of both investments in new machines and maintenance services. We end up by analyzing the effects of technological progress on optimal plans and prove that a negative anticipation effect can occur even without any market imperfections.

*The first author was financed for this research by the European Commission under the RTN Grant project No. HPRN-CT-000234-2002. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the European Commission. The second author was partly supported by the Austrian Science Foundation under contract 15618, and by the Austrian National Bank (ÖNB) under grant No. Jubiläumsfond 11260 ("Optimal Investment in Technology under Adverse Investment Effects on Sales").

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Keywords: Vintage capital, embodied technological progress, investment analysis, maintenance services, substitution vs complementarity.

JEL Classification: D92, O33, E22, C61.

1 Introduction

The need to introduce heterogeneity in the capital-accumulation process and to differentiate capital goods by their vintage or productivity have been widely recognized since the seminal work of Solow [37]. Together with embodied technological progress, the most crucial aspect of the vintage capital models is that capital goods of later date are more productive, or make products of higher quality (see [28], [38], [2] and [7]). Accordingly, recent studies (see among others, [40], [16], [1], [17]) take into account the fact that firms not only have to decide on the volume of the capital goods but also on the optimal age distribution of them, and have been centered around the questions posed by Chari and Hopenhayn [12]: Why are new technologies often adopted so slowly? Why do people often invest in old technologies even when apparently superior technologies are available? How are decisions to adopt new technologies affected by the prospect that even better technologies will arrive in the future?

Vintage capital models are able to generate different properties and dynamics from the classical capital accumulation models. The so-called "*vintage effect*", i.e. the productivity differential between successive vintages of capital due to embodied technological progress, plays a crucial role on the firm's optimal plans. Such a productivity differential creates an advantage in investments in younger machines. Younger machines are not only endowed with a superior technology but have also a longer lifetime than the older ones (see [5]). However, they have also the disadvantage that the older machines are cheaper and the costs of depreciation and discounting are less. With the presence of these latter effects, [16] provides an explanation for why new technologies are often adopted on a large scale only after a long period of time. Taking into account the vintage effect, [16] analyzes in what way a perfectly competitive firm adjusts current investments to the predictions of technological progress. As current investments do not affect the profitability of investments in future technologies in a perfectly competitive market, predictions of higher technological progress in the future do not influence the current investments. However, considering market power, [17] shows that a "*negative anticipation effect*" occurs. Since current investments increase output which decreases the price, this creates a negative effect on the profitability of future investments, so that the anticipated technology shocks will be preceded by declines of investment. This will be followed by a period of higher growth where new capital goods can be purchased without reducing the output price too much.

A common assumption in these studies and also in most of the macroeconomic literature is that the depreciation rate is either exogenously fixed or acts as a residual (*depreciation-in-use hypothesis*) independent of maintenance activities. However, empirical (see among others, [23], [32], [30], [25]) and theoretical (see [20], [33], [9]) findings state that depreciation rate is neither constant, as it increases with the ageing of the capital stock, nor a residual variable as it can be controlled by the economic agents by choosing appropriate levels of maintenance expenditures. Indeed, many firms have in mind the maintenance implications of their adoption decisions. Even a firm can disregard the adoption of a new technology if it anticipates a costly pace of maintenance costs. As argued by McGrattan and Schmitz [30], maintenance expenditures are too important to be neglected even at the aggregate level and hence, become inevitable when analyzing how the firms set their optimal plans. The endogenous nature of depreciation rate depending on maintenance expenditure also highlights the trade-offs between new investment and higher maintenance expenditures and induces the following questions to emerge in order to reach a better understanding of the firm's capital accumulation process:

i) Under what conditions do maintenance and investment appear as substitutes or complements?

ii) What is the optimal allocation of investment and maintenance across vintages?

iii) How do maintenance and investment decisions shift in response to an exogenous change in the rate of technological progress?

iv) Can there be a negative anticipation effect even under non-monopolistic settings?

The economic literature devoted to the role of maintenance in the economy is mostly composed of empirical or computational studies concerned with the cyclical properties of maintenance and its implications for the business cycle (see among others [29] and [14]. Among very few theoretical contributions, [6] studies the optimal allocation of labor resources to production, technology adoption or capital maintenance in a one-hoss-shay vintage capital model and points out that though capital maintenance deepens the technological gap by diverting labor resources from adoption, it generally increases the long run output level at equilibrium. In a very recent study, how technological progress both embodied and disembodied affects the life-time of capital have been analyzed by [4]. Apart from these studies concerned with the economic performances at the aggregate level, there is no accompanying theoretical contribution in a vintage capital framework at the firm level. [26] and [9] investigate the demand for maintenance services in some typical firms' investment problems with a deliberate microeconomic approach. However, both studies do not take into consideration the ageing of the capital stock and can not attempt to answer the questions (ii) and (iv).

This paper provides the needed analysis of the firm's capital accumulation process taking into account that depreciation is endogenous and in particular associated with vintage specific maintenance expenditure. We present precise answers to the questions (i)-(iv) with a detailed analysis of the substitutability between investment and maintenance services. We provide an intrinsic definition of complementarity vesus substitutability and prove that maintenance is a local substitute for investment as soon as the marginal cost of maintenance is strictly increasing. We show that maintenance and investment in new capital goods appear as complements with respect to the changes in productivity, cost of maintenance, fixed cost of operation and the efficiency of maintenance services. We also find that investment in new capital goods and maintenance services are substitutes in the traditional sense: when the price of new machines changes, the demands for capital goods and for maintenance respond in the opposite direction. We analyze the effects of technological progress on the firm's optimal plans and prove that a negative anticipation effect can occur even without any market imperfections. Our set up allows for an extension where investments in the old vintages are possible. We show that investment in old machines appears as a substitute of both investments in new machines and maintenance services.

A remarkable feature of our analysis is that it does not rely on particular parameterizations of the exogenous functions involved in the model, rather, it uses only general and plausible qualitative properties. Therefore the obtained results are robust.

The paper is organized as follows. Section 2 describes the model. Section 3 presents the optimality conditions and the main assumptions of the analysis. In Section 4 we conduct comparative economic analysis and aim to answer whether investments and maintenance appear as complements or substitutes with respect to the variations in different exogenous factors. We also make the analysis under an intrinsic definition of substitutability. Section 5 analyzes the negative anticipation effect of the technological progress and presents numerical illustrations. Section 6 extends the model to allow for invests in old machines and discusses its implications. Some more technical proofs are given in Appendix.

2 The Model

In this section we present our model, which has the form of an age-structured PDE optimal control system with the main dynamic equation similar to that introduced in [1]. The time is continuous, and for convenience we consider a finite (but "large") horizon $[0, T]$. For any time t, we denote by $K(t, a)$, the number of machines[1] of age a which are in use by the firm. Each machine has a fixed maximal lifetime ω. The productivity of the machines build at time t is $f(t)$, where it is assumed that due to *embodied technological progress*, the new generations of machines are more productive than the old ones, that is, f is increasing. Following [16], the total output produced at time t by the firm is defined as

$$Q(t) = \int_0^\omega f(t-a)K(t,a)\,\mathrm{d}a,$$

where $f(t-a)$ is the productivity of the machines that at time t are of age a, that is, the productivity of the machines of technology vintage $t-a$. The firm's revenue, is then $pQ(t)$, where p is the output price[2].

When describing the evolution of the capital stock we take into account the dependence of the capital depreciation on age and on maintenence. The depreciation is typically larger for old capital goods (see for example, [8], [17]). Moreover, it is far from convincing that the capital depreciation rate is exogenous, especially at the firm level (see [30], [9]). Firms typically control the depreciation rate of capital by choosing an appropriate level of vintage-specific maintenance services. Therefore we consider the level of maintenance services that the firm chooses for the machines of age a at time t, denoted by $m(t, a)$, as a control variable. It is clear that the more are the resources devoted to maintenance, the less will be the depreciation rate. With these in mind, the evolution law of capital stock is described by

$$K_t + K_a = -\delta(a, m(t,a))K(t,a), \quad K(0,a) = K_0(a), \quad K(t,0) = I(t),$$

where the subscripts denote the partial differentiation, $I(t)$ is the inflow of new machines that the firm purchases at time t, $K_0(a)$ is the given initial data for the firm's capital, and δ is the physical depreciation rate, depending on the maintenance level. In the present basic version of the model we do not include the possibility to invest in non-frontier vintages, for the sake of simplicity. Section 6 discusses some consequences of this possibility. The strict meaning of the solution of the above equation is given e.g. in [19] or [42].

We denote by $C(I)$ the total (acquisition and implementation) cost of installing I new machines. Although typically $C(I)$ is a linear-quadratic function, in our analysis it may have

[1] Here "number" is a colloquial term. Strictly speaking, $K(t, \cdot)$ is a (non-probabilistic) density, so that $\int_{a_1}^{a_2} K(t,a)\,\mathrm{d}a$ is the number of machines of age between a_1 and a_2.

[2] An output-dependent price $p = p(Q(t))$ could also be investigated in the same framework, following [17]. However, in this paper we assume perfect competition in order to focus on the effects caused by the maintenance, rather than of the output dependence of price, which is investigated in the abovementioned paper.

a rather general form. Notice that $C(I)$ is assumed independent of time, which means that although the newer machines become more productive due to the technological progress, the acquisition cost remains the same. A typical example is the computer industry (see [27]).

The cost of maintenance services is represented by $D(M)$ where M accounts for the total maintenance services at time t,

$$M(t) = \int_0^\omega (m_0 + m(t,a))K(t,a)\,\mathrm{d}a.$$

under the fact that as long as a machine is in operation it requires a fixed maintenance (or operation) cost, $m_0 \geq 0$, in addition to the controlled maintenance level $m(t,a)$ which is a matter of the firm's choice.

The firm maximizes the discounted value of the cash flow over the planning horizon $[0,T]$. Denoting the discount factor by r, we obtain the following age-specific dynamic optimization model of the firm:

$$\max_{I,m} \int_0^T e^{-rt}\left[pQ(t) - C(I(t)) - D(M(t))\right]\mathrm{d}t \tag{1}$$

subject to

$$K_t + K_a = -\delta(a, m(t,a))K(t,a), \quad K(0,a) = K_0(a), \quad K(t,0) = I(t), \tag{2}$$

$$Q(t) = \int_0^\omega f(t-a)K(t,a)\,\mathrm{d}a, \tag{3}$$

$$M(t) = \int_0^\omega (m_0 + m(t,a))K(t,a)\,\mathrm{d}a, \tag{4}$$

$$m(t,a) \geq 0, \tag{5}$$

$$I(t) \geq 0. \tag{6}$$

Standing assumptions:

(i) C, $D : [0,\infty] \mapsto [0,\infty]$ have continuous second derivatives, and $C(0) = 0$, $C'(I) > 0$, $C''(I) \geq \gamma^C > 0$, $D(0) = 0$, $D'(M) \geq \gamma^D > 0$, $D''(M) \geq 0$;

(ii) $\delta : [0,\omega] \times [0,\infty] \mapsto [0,\infty]$ is twice differentiable in m with Lipschitz continuous δ'_m and δ''_{mm}, and $\delta'_m(a,m) < 0$, $\delta''_{mm} > 0$;

(iii) $f : [-\omega, T] \mapsto (0,\infty)$ is piecewise continuous and non-decreasing;

(iv) $K_0 : [0, \omega] \mapsto [0, \infty)$ is continuous, $m_0 \geq 0$, $r \geq 0$, $p > 0$.

We stress that the principle assumptions, namely that C and δ are strongly convex (the second one with respect to m) are plausible. The first one is standard, while the second means that the marginal efficiency of maintenance decreases. For the function

$$\delta(a, m) = \bar{\delta}(a) + \frac{1}{\alpha + \beta m}, \quad \alpha,\, \beta > 0, \tag{7}$$

considered in [9], for example, assumption (ii) is apparently fulfilled. Here $\bar{\delta}(a)$ (presumably increasing) is the lower bound for the depreciation rate at age a, while $\bar{\delta}(a) + 1/\alpha$ is the depreciation rate without maintenance.

Proposition 1 *The problem (1)–(6) has a solution.*

The proof (which is not obvious) is sketched in Appendix. We stress that the solution for m need not be unique. Indeed, if for some t and a it happens that the optimal $K(t, a) = 0$ then the value of $m(t, a)$ does not matter. At such points (if any) we set $m(t, a) = 0$.

3 Optimality Conditions

In order to solve the problem (1)–(6) we apply the maximum principle for general age-structured optimal control problems obtained in [19]. We mention that the earlier optimality conditions (see [10]) do not fit to the problem due to the presence of the control m in both the differential equation (2) and the integral expression (4). The maximum principle in [19] implies the following.

Proposition 2 *Let I, m, K, Q, M be an optimal solution of problem (1)–(6). Then the adjoint equation*

$$\xi_t + \xi_a = (r + \delta(a, m(t, a)))\xi - pf(t - a) + D'(M(t))(m_0 + m(t, a)), \tag{8}$$
$$\xi(t, \omega) = 0, \quad \xi(T, a) = 0$$

has a unique solution ξ, and

$$\xi(t, 0)I(t) - C(I(t)) = \max_{I \geq 0}\{\xi(t, 0)I - C(I)\} \quad \textit{for almost all } (t, a), \tag{9}$$

$$\xi(t, a)\delta(a, m(t, a)) + D'(M(t))m(t, a) = \min_{m \geq 0}\{\xi(t, a)\delta(a, m) + D'(M(t))m\} \tag{10}$$
$$\textit{for almost all } (t, a) \textit{ for which } K(t, a) \neq 0.$$

Since the optimal solution in the interval $[0, \omega]$ strongly depends on the initial data $K_0(a)$ (which are in a sense arbitrary), and in the interval $[T - \omega, T]$ is strongly distorted by the forthcoming end of the planing horizon, the economically meaningful analysis in the next section will be restricted to the time interval $[\omega, T - \omega]$.

The following assumption and lemma allow to give a more explicit representation of the optimal controls from the maximum principle.

Regularity assumption:
The following inequality is fulfilled for the optimal maintenance costs:

$$pf(t - \omega) > D'(M(t))m_0 \quad \text{for all } t \in [0, T]. \tag{11}$$

Inequality (11) means that the marginal revenue obtained from the oldest machines in use exceeds the marginal expenditure of maintaining the machines at the minimal required level.

Lemma 1 *For the solution ξ of the adjoint equation (8) it holds that $\xi(t, a) > 0$ for every $t \in (\omega, T)$ and $a \in [0, \omega)$.*

Thanks to the above lemma the optimality conditions (9) and (10) can be rewritten in the following more convenient form. Define the functions

$$(C')_+^{-1}(y) = \begin{cases} (C')^{-1}(y) & \text{if } y \geq C'(0), \\ 0 & \text{else,} \end{cases}$$

$$(\delta'_m)_+^{-1}(a, y) = \begin{cases} (\delta'_m)^{-1}(a, y) & \text{if } y \geq \delta'_m(a, 0), \\ 0 & \text{else,} \end{cases}$$

where $(C')^{-1}$ and $(\delta'_m)^{-1}$ denote the inverse functions of the corresponding derivatives (with respect to m for δ'_m). Then the optimal investment and maintenance can be expressed from (9) and (10) as

$$I(t) = (C')_+^{-1}(\xi(t, 0)), \qquad m(t, a) = (\delta'_m)_+^{-1}\left(a, -\frac{D'(M(t))}{\xi(t, a)}\right). \tag{12}$$

Remark 1 It is useful to notice that in (12) $(C')_+^{-1}$ is increasing with respect to $\xi(t, 0)$ and $(\delta'_m)_+^{-1}$ is increasing with respect to $\xi(t, a)$. The assumptions for C, D, and δ imply that at points where I (resp. m) is positive, the increase is strict.

Let us denote

$$\hat{m}(a, M, \xi) = (\delta'_m)_+^{-1}\left(a, -\frac{D'(M)}{\xi}\right). \tag{13}$$

With this feed-back low the adjoint equation becomes

$$\xi_t + \xi_a = (r + \delta(a, \hat{m}(a, M(t), \xi)))\xi - pf(t-a) + D'(M(t))(m_0 + \hat{m}(a, M(t), \xi)). \tag{14}$$

The right-hand side is well-defined and Lipschitz continuous in $\xi \geq 0$. Indeed, if ξ is close to zero, then $-\frac{D'(M)}{\xi} < \delta'_m(a, 0)$ due to the assumption that $D'(M) \geq \gamma^D > 0$, hence $\hat{m}(a, M, \xi) = 0$. The other points of non-differentiability of $\hat{m}$ in ξ are caused by the operation of maximum involved in $(\delta'_m)_+^{-1}$, which does not spoil the Lipschitz continuity. In particular, (14) has a unique solution, which necessarily coincides with the one of (8).

4 Comparative economic analysis

In this section we investigate how the optimal investment and maintenance depend on different exogenous factors involved in the model. In some of the considerations below we assume that the maintenance cost, $D(M)$, depends linearly on the total maintenance and operation services, that is $D(M) = dM$ ($d > 0$ is a constant). This is a substantial simplification, since in this case the adjoint equation (14) completely decouples from the state equation and can be investigated separately. Namely, it takes the form

$$\xi_t + \xi_a = (r + \delta(a, \hat{m}(a, \xi)))\xi - pf(t-a) + d(m_0 + \hat{m}(a, \xi)), \quad \xi(t, \omega) = 0, \quad \xi(T, a) = 0, \tag{15}$$

where now we skip the argument $M(t)$ in the notation $\hat{m}(a, M, \xi)$, since the latter is independent of M. As Remark 3 shows, the assumption for linearity of the maintenance cost is essential for the validity of the next proposition. However, the two most interesting statements below are proved in the case of a general nonlinear maintenance cost function. In what follows it is supposed that the standing and the regularity assumptions hold for the sets of data involved in the considerations. Also the linearity condition $D(M) = dM$ is assumed in this section, unless an alternative condition is specifically formulated.

Proposition 3 *Consider two technology functions f_1 and f_2 for which $f_1(t) < f_2(t)$ at a time $t \in [0, T - \omega]$. Then for the corresponding optimal solutions, (I_1, m_1) and (I_2, m_2), it holds that*

$$I_1(t) \leq I_2(t), \quad m_1(t + a, a) \leq m_2(t + a, a) \;\; \textit{for every } a \in [0, \omega].$$

Moreover, each of the inequalities is strict, unless $I_1(t) = I_2(t) = 0$ or $m_1(t, a) = m_2(t, a) = 0$, respectively.

According to this proposition, investment and maintenance appear as complements with respect to the productivity. It is optimal for a firm to invest more in technologies of higher productivity and also to maintain more these vintages.

The proof of the above proposition is simple, therefore we present it in the main text.

Proof of Proposition 3. On every characteristic line starting at the point $(t, 0)$ (that is on the segment $\{(t+s, s) : \ s \in [0, \omega]\}$) the solution ξ of the adjoint equation (15) satisfies an ordinary differential equation. Namely, $\eta(s) = \xi(t+s, s)$ satisfies

$$\eta'(s) = (r + \delta(s, \hat{m}(s, \eta(s)))\eta(s) - pf(t) + d(m_0 + \hat{m}(s, \eta(s))), \tag{16}$$

with $\eta(\omega) = 0$. We have to compare the solutions η_1 and η_2 of this equation with f_1 and f_2 substituted for f. If $\eta_1 = \eta_2$, then the difference of the right-hand sides is

$$\eta_2' - \eta_1' = -p(f_2(t) - f_1(t)) < 0.$$

Since $\eta_1(\omega) = \eta_2(\omega) = 0$, this implies that $\eta_2(s) > \eta_1(s)$ for $s \in [0, \omega)$. Then the formulas in (12) and Remark 1 imply the claims of the proposition. Q.E.D.

Corollary 1 *Let $\bar{a}_i(t)$ ($i = 1, 2$) be the maximal age of maintaining machines at time t for the technology function f_i, that is, $m_i(t, a) = 0$ for $a \geq \bar{a}_i(t)$, but $m_i(t, a - \varepsilon) > 0$. Assume that $\bar{a}_2(t) > 0$. Then*

$$\bar{a}_1(t) < \bar{a}_2(t).$$

The maximal age of the machines to be maintained would be higher for the more productive technologies. Even if it may be technologically feasible to maintain a machine till the end of its maximal life-time, it ceases to make economic sense at some point because the upkeep ultimately becomes more expensive compared with the cost of a new and superior machine (see [24]). This sheds some light on the rapid disappearance of early models in the computer industry. As pointed out by [21], at an age of one year, the RAM of a used computer is 48 percent below the median RAM of a new computer, its speed is 36 percent slower and its hard disk is 52 percent smaller. For older ages, the decline is rapid and continues to fall but the oldest ages. The new computer models are typically more productive, therefore they are maintained more relative to the older ones.

Proposition 4 *Consider two prices of maintenance, $d_1 < d_2$. Then for the corresponding optimal solutions, (I_1, m_1) and (I_2, m_2), it holds that*

$$I_1(t) \geq I_2(t), \quad m_1(t, a) \geq m_2(t, a) \ \textit{for every } a \in [0, \omega].$$

Moreover, each of the inequalities is strict, unless $I_1(t) = I_2(t) = 0$ *or* $m_1(t, a) = m_2(t, a) = 0$*, respectively.*

The proof of this proposition uses exactly the same argument as the previous one (excepting the case $m_0 = 0$, which requires some more routine work), therefore we skip it.

Corollary 1 applies also to Propositions 4. Exactly the same claim as in Proposition 4 holds for the dependence of the optimal solution on the fixed cost of operation, m_0.

Investment and maintenance appear as complements with respect to the price of maintenance and with respect to the fixed costs of operation. Remembering the empirical assesment of the economy-wide importance of the maintenance costs in [30] (around 6% of the Canadian GDP), this proposition becomes vital in exploring the firm's optimal plans analytically. An increase in the price of maintenance services leads firms to lower their demand for both investment goods and maintenance services. Thus, in accordance with [9], instead of finding that maintenance is a substitute for investment expenditures, we found that investment and maintenance behave as gross complements. This confirms the empirical findings of [31] that the investment has a positive and statistically significant impact on the maintenance spending. The interpretation is that investment acts as a proxy for demand conditions that exert a direct impact on the desire to maintain existing stock. An increase in the price of maintenance leads to a decrease in both the maintenance and the investment activities due to an income effect and leads to a decrease in the maximal age of the machines to be maintained due to a substitution effect. These results reinforce the empirical findings of [34] that puts forward the increase in the relative cost of maintenance as the source of rising scrapping rates between 1947 and 1969 on the domestically produced post-war vintage automobiles in United States.

According to Proposition 4, investment and maintenance appear as complements also with respect to the fixed costs of operation. Accordingly, the maximal age of the machines to be maintained decreases with the increasing fixed costs of operation and with the price of maintenance. It is important to note that the maintenance is lower when the cost of operation is higher. The fixed cost assumption is motivated by empirical observations. For instance, the increase in demand for IT positions is a clear indication of the need to backup computer investments with outlays on maintenance and support. Indeed, research by Gartner Group (1999), a private consulting firm shows that, as of 1998, for every $1 that firm spent on computers there was another $2.30 spent on wages for IT employees and consultants (see [43]). The fixed cost of operation induces the phenomenon of obsolescence. Once the marginal productivity of a machine falls behind the fixed cost of operation the firm will choose not to maintain the machine anymore. These results allow to explain why older models disappear more rapidly from the market in computer industry especially in comparison with the auto industry. One car for \$50 000 is operated by one person, one PC for \$1 000 - by one person. If the salary of the two workers are about the same which are included in the cost of operation, then m_0 for computers is 50 times larger than that for cars (per input price) leading to a lower level of maintenance and thus higher depreciation rates for the computers.

Proposition 5 *Consider two depreciation functions δ_1 and δ_2 for which the standing assumptions hold, and additionally δ_1 and δ_2 are related in the following way: for every $a \in [0,\omega]$ and $m \geq 0$ (i) $\delta_1(a,m) \leq \delta_2(a,m)$; (ii) $\delta'_{1,m}(a,0) \leq \delta'_{2,m}(a,0)$, where $\delta'_{i,m}$ is the derivative of δ_i with respect to m. Then for the corresponding optimal solutions, (I_1, m_1) and (I_2, m_2), it holds that*

$$I_1(t) \geq I_2(t), \quad m_1(t,a) \geq m_2(t,a) \text{ for every } a \in [0,\omega]. \tag{17}$$

Moreover, if the inequality in (i) is strict for $a > 0$, then each of the inequalities (17) is also strict, unless $I_1(t) = I_2(t) = 0$ or $m_1(t,a) = m_2(t,a) = 0$, respectively.

The proof uses once again a comparison argument for equation (16), but requires some more technical arguments, therefore it is presented in Appendix.

Investment and maintenance appear to be complements also with respect to the efficiency of maintenance. This means that higher depreciation rate for the same maintenance level leads to less investment and maintenance. However, the situation is different for the dependence of the optimal investment and maintenance on the acquisition cost of machines.

Proposition 6 *Consider two costs functions for new capital, C_1 and C_2, where $C'_1(I) < C'_2(I)$ for all $I > 0$. Then for the corresponding optimal solutions, (I_1, m_1) and (I_2, m_2), it holds that*

$$I_1(t) \geq I_2(t), \quad m_1(t,a) = m_2(t,a) \text{ for every } a \in [0,\omega].$$

Moreover, the inequality is strict, unless $I_1(t) = I_2(t) = 0$.

For a proof it is enough to notice that the adjoint equation (15) does not depend on the function C, therefore the formulae for I and m in (12) imply the claim.

Remark 2 The independence of the maintenance on the price of the new machines is heavily bind with the assumption of linearity of the maintenance cost. If for the cost-of-maintenance function, $D(M)$, it holds that $D''(M) \geq \rho > 0$ for every $M \geq 0$, then

$$m_1(t,a) < m_2(t,a) \text{ for some } (t,a) \in [\omega, T-\omega] \times [0,\omega],$$

provided that $m_2(t,a)$ is not identically zero. The proof of this claim is much more involved and we do not present it.

The decision on whether to maintain an existing machine or letting it scrap has to be made using the marginal costs of new machines. This issue has been the subject of many empirical studies (see, among others, [36], [35], [3], [32]). In a recent paper, [25] disentangles the source of the dramatic increase in the longevity of automobiles over the past 25 years and concludes that with the increase in the new car prices and the decrease in the expense of auto maintenance, it is optimal to increase the maintenance to an older age. However, being among the very few analytical contributions, [9] with a homogeneous stock of capital goods assumption, mentions that maintenance and investment act as complementary even with respect to the acquisition cost of machines. In this respect, Proposition 6 together with Remark 2 analytically confirm the empirical findings and prove to be crucial in identifying the effect of the acquisition costs of machines on the scrapping rates and the link between investment and maintenance. In contrast to [9], allowing for differentiation of capital goods, our model confirms that maintenance and investment appear to be substitutes with respect to the acquisition cost of machines when one assumes a convex cost of maintenance services. Accordingly, the increase in the price of new machines should lead firms to delay replacement and to increase maintenance, thus reducing the rate of depreciation.

In order to make the further analysis clearer we show that under an additional natural assumption (which, however is not used in Proposition 7 below) the maintenance of machines of certain vintage decreases with the age.

Lemma 2 *Assume that the productivity function f is differentiable, that the depreciation function $\delta(a,m)$ is differentiable in a, that δ_m is independent of a, and that $\delta'_a(a,m) \geq 0$ (alder machines depreciate faster). Then for the optimal maintenance m and every $t \in [0, T-\omega]$ and $a \in (0,\omega)$ for which $m(t,a) > 0$ it holds that*

$$\frac{\partial m}{\partial a}(t,a) \leq 0.$$

The proof is given in Appendix. Notice that the particular function δ defined in (7) apparently satisfies the assumptions of the lemma if $\bar{\delta}$ is differentiable and increasing.

Now we investigate the impact of the *rate of the technological progress* on the drop of maintenance with age. In order to filter out the role of the other factors, in particular of the productivity level, we consider two identical firms, differing only in the rate of the technological progress of the equipment they use. For the corresponding (differentiable) technology functions, f_1 and f_2 we assume that at a time $t^* \in [0, T-\omega]$ it holds that $f_1(t^*) = f_2(t^*)$ (that is, the firms are competitive), while $f_1'(t^*) < f_2'(t^*)$. Denote by m_1 and m_2 the corresponding optimal maintenance functions.

Proposition 7 *For every $a \in (0, \omega)$ for which $m_1(t^* + a, a) > 0$ it holds that*

$$\frac{\partial m_1}{\partial a}(t^* + a, a) > \frac{\partial m_2}{\partial a}(t^* + a, a). \tag{18}$$

To avoid misunderstanding we stress that the functions m_1 and m_2 are first differentiated in their second argument, a, and then the derivatives are evaluated at $(t^* + a, a)$. The proof of the proposition indicates how the above inequality can be interpreted if some of the derivatives in (18) does not exist.

Proof. Consider ages $a_1 = a - h$ and $a_2 = a + h$, where $h > 0$ is a small increment. Due to the assumptions that $f_1(t^*) = f_2(t^*)$ and that $f_1'(t^*) < f_2'(t^*)$ we have that $f_1(t^* - h) > f_2(t^* - h)$ for small $h > 0$. Similarly, $f_1(t^* + h) < f_2(t^* + h)$. Then, according to Proposition 3, for any $a \in (0, \omega)$ it holds that

$$m_1(t^*+a, a+h) = m_1((t^*-h)+(a+h), a+h) > m_2((t^*-h)+(a+h), a+h) = m_2(t^*+a, a+h)$$

and similarly $m_1(t^* + a, a - h) < m_2(t^* + a, a - h)$ (the inequalities are strict where some of the m-s are positive). Then

$$m_1(t^* + a, a + h) - m_1(t^* + a, a - h) > m_2(t^* + a, a + h) - m_2(t^* + a, a - h).$$

Dividing by h and passing to the limit (where we should notice that the strict inequality is uniform in h) we complete the proof. Q.E.D.

Embodied technological progress is both quantitatively and qualitatively one of the most important features of investment dynamics (see [22]). As mentioned in [4], a technological acceleration induces two opposite effects: an incentive to reduce maintenance and scrap earlier in order to profit from the increased efficiency of new vintages but also an incentive to increase maintenance and delay scrapping due to a drop in the profitability of investment. The above proposition resolves this trade-off analytically so that the maintenance appears to be a substitute of technological growth. Accordingly, depreciation rate is an increasing function of embodied technological progress and this implies a strong mechanism through which embodiment affects capital depreciation. In particular, one can put forward technological acceleration as the main source of rapid disappearance of early models in the computer industry. It is well known that the surge in computer investment has come as a direct result of rapid price declines which themselves have been due to rapid technological progress (see among others [43], [44], [21]). Trying to explain how durables with high depreciation rates may have more volatile expenditures, [39] presents that car bodies are redesigned every 4-5 years and new generations of Intel processors appear on average 2-3 years and computers have higher depreciation rates compared to autos. Indeed, [15] measures that the quality adjusted price has been falling at an average rate of 23% for computers and 2.5% for autos a year during 1960-2000. In confirmation with these, as the technological progress in computer industry has been much more rapid, it is clear from Proposition 7 that the resources devoted to maintenance would be lower leading to higher scrapping rates.

So far we investigated whether investments and maintenance appear as complements or substitutes with respect to variations in different exogenous factors. Now we give an intrinsic definition of complementarity versus substitutability, which is related to the terminology in [30]. In contrast to the previous contributions which are mainly empirical, we present a full analytical characterization aiming to find out to what extent maintenance can be a substitute for investment.

Definition 1 Let I and m be the optimal investment and maintenance functions in problem (1)–(6), and let $V(I, m)$ be the corresponding objective value. Maintenance is called a *local intrinsic substitute (complement)* of investment, if for any smaller investment function $\tilde{I}(t) \leq I(t)$, $t \in [0, T]$ (where the inequality is strict whenever $I(t) > 0$), it holds that for every maintenance function $\tilde{m}$ for which $\tilde{m}(t, a) > m(t, a)$ ($\tilde{m}(t, a) < m(t, a)$, respectively) whenever $m(t, a) > 0$, it holds that

$$V(\tilde{I}, m + h(\tilde{m} - m)) > V(\tilde{I}, m)$$

for all sufficiently small $h > 0$.

Thus m is a local substitute for I, if for any down-ward shifted investment function $\tilde{I}$ (clearly $V(\tilde{I}, m) \leq V(I, m)$), it holds that each sufficiently small up-ward shift of m improves the objective value. Certainly it may happen that maintenance is neither a substitute nor a complement of investment. However, under natural conditions it could be argued that m cannot be both substitute and complement for I (in particular, this follows from the proof of the next proposition).

Proposition 8 *Assume that $I(t) > 0$ for every $t \in [0, T - \omega)$, and that $m(t, a)$ is not identically zero. Assume also that the marginal cost of maintenance is strictly increasing, that is $D''(M) \geq \rho > 0$. Then maintenance is a local intrinsic substitute for investment.*

The proof is given in Appendix.

5 Anticipation effects of the technological progress

Our aim in this section is to contribute to the answer of the following question formulated by [12]: *How are decisions to adopt new technologies affected by the prospect that even better technologies will arrive in the future?* This question is increasingly important if one has in mind the recent dynamics in the computer industry. Following from Figure 5 in [39], it is clear that the general pattern seems to be one-year expenditure spikes (in 1982, 86, 91 and 95) followed by several years of falling investment rates. Apparently, investments are optimally synchronized with the arrival of the each new operating system. Although many similar examples from different sectors can be found, the theoretical analysis of such an anticipation effect and the conditions under which it exists are limited in the literature.

In a perfectly competitive environment, [16] have shown that the future technological developments have no effect on current investment. The firm is assumed to be too small to influence output prices implying that the revenues of different machines do not influence each other so that current investments do not affect the profitability of investments in future technologies. On the contrary, as shown in [17], the situation changes if the firm has a market power. A negative anticipation effect occurs so that current investments in recent generations of capital goods decline when faster technological progress is expected to take place in the future. In addition to this, it is also proven in [18] that a negative anticipation effect of the technological progress can prevail due to imperfect financial markets. However, all of these studies assume an exogenous rate of depreciation and neglect the importance of the maintenance activities. With these in mind, a natural question arises: can an anticipation effect take place even in a perfectly competitive environment without imperfections, by taking into account the endogenous nature of the age-specific depreciation rate?

Our goal is to analytically investigate how an expected technological breakthrough influences the optimal investment and maintenance of a firm. Following [17] we consider the following two technology functions f:

$$f(s) = f_1, \quad \forall s \in [0, T], \tag{19}$$

and

$$\bar{f}(s) = \begin{cases} f_1 & \text{for } s \le \bar{t}, \\ f_2 & \text{for } s > \bar{t}, \end{cases} \tag{20}$$

where $\bar{t} \in (\omega, T - \omega)$ is a time of a technological breakthrough[3], thus $f_2 > f_1$.

We shall compare the optimal behavior of two identical firms assuming that one of the firms anticipates the technological breakthrough at time $\bar{t}$ from the very beginning of the planning horizon (thus it solves its optimization problem with the technology function (19)), while the second firm learns about the technology shock when it occurs (thus its behaviour before time $\bar{t}$ is determined by the technology function (20)).

Denote by I, m, K, Q, M the solution of optimization problem (1)–(6) for the non-anticipating firm (with technology function $f(s) = f_1$), and by $\bar{I}$, $\bar{m}$, $\bar{K}$, $\bar{Q}$, $\bar{M}$ – the optimal solution for the anticipating firm (with technology function $\bar{f}$).

Our main result in this section is preceded by two lemmas which are essential for the proof but also are of economic interest.

The first one claims continuity of the maintenance costs $\bar{M}(t)$ and $M(t)$. We mention that the continuity is not an obvious fact, since, as it will be seen in the second lemma, the investment, $\bar{I}$, and the maintenance services, $\bar{m}$, of the anticipating firm are discontinuous if $f_2 > f_1$.

Lemma 3 *The functions $\bar{M}$ and $\bar{Q}$ are Lipschitz continuous.*

Lemma 4 *(i) The optimal investment, $\bar{I}$, is Lipschitz continuous on $[\omega, T - \omega]$, excepting the moment $t = \bar{t}$ where $\bar{I}$ has an upward jump of magnitude at least $c.(f_2 - f_1)$ (c is a positive constant independent of $f_1 < f_2$, with a bounded f_2).*
(ii) The optimal maintenance, $\bar{m}$, is Lipschitz continuous, excepting the segment $\{(\bar{t}+s, s),\ s \in [0, \omega)\}$, where it has an upward jump of magnitude at least $c.(\omega - s).(f_2 - f_1)$ (for those s for which $\bar{m}(\bar{t} + s - 0, s - 0) > 0$).

The next proposition claims that the presence of maintenance costs leads to a decrease in the investments (negative anticipation effect) before an expected technological breakthrough. To simplify the proof we assume that the maintenance level $m(t, a) \geq 0$ is fixed and the optimization in (1)–(6) is carried out only with respect to the investments $I(t)$.

Proposition 9 *Assume that the fixed operation cost, m_0, is positive, and that the maintenance cost function is marginally increasing: $D''(M) \geq \rho > 0$. Then $\bar{I}(t) < I(t)$ in some interval contained in $(\bar{t} - 2\omega, \bar{t})$.*

[3]As before, we exclude from the consideration the initial and the final intervals of length ω since the behaviour of the firm in these intervals is strongly influenced by the initial data and by the end of the planning horizon, respectively.

The proofs of the above lemmas and proposition are technical and are presented in Appendix.

Our analysis shows that a negative anticipation effect occurs even for a perfectly competitive firm operating in a perfect financial market if the firm has the freedom to chose the level of maintenance of its machines (hence the scrapping time), under the assumption that the maintenance costs are marginally increasing. As the next experiments show, the negative anticipation effect holds not only for the investment, but also for the maintenance (if used as a second control variable): the anticipating firm maintains the machines produced before the technological breakthrough less than the non-anticipating one, and this drop in maintenance begins even before the breakthrough.

Figures 1 and 2 represent the optimal investment and maintenance for the anticipating firm with the following data specifications, which are economically plausible[4]:

$p = 1$;
$\omega = 10$;
$r = 0.003$;
$m_0 = 0.6$;
$C(I) = bI + 0.5cI^2$, with $b = 5$, $c = 1$;
$D(M) = b_m M + 0.5c_m M^2$, with $b_m = 0.6$, $c_m = 0.03$;
$\delta(a, m) = \bar{\delta} + 1/(\alpha + \beta m)$, $\bar{\delta} = 0.02$, $\alpha = 2$, $\beta = 8$;
$\bar{f}(t)$ - given by (20) with $\bar{t} = 30$, $f_1 = 20$, $f_2 = 30$.

[Figure 1 about here.]

[Figure 2 about here.]

Figure 1 shows the drop of investments before the technological breakthrough at time $\bar{t} = 30$ and the jump up at $\bar{t}$. The solid line corresponds to the investment level of the non-anticipating firm before $\bar{t}$. Just before the technological break-through the firm reduces the investments in new machines in order to invest more after the break-through as it becomes more profitable to wait for the new generations endowed with more efficient technologies. It should be noted as well that the steady level of investment after the shock is higher than that before the shock. The same result applies to the maintenance activities as well, reinvoking the importance of Propositions 3–5. Figure 2 represents the

[4]The numerical solution of the of the problem (1)–(6) and of the extended problem considered in the next section are obtained by the general solver developed by the second author, which is presented in [13, 41].

maintenance services, which also decrease for machines of the old technology even before $\bar{t}$, and jump up for the machines of the new technology (produced after $\bar{t}$). It is optimal for the firm to reduce maintenance and let the older vintages scrap earlier just before the shock in order to be able to devote these resources for maintaining the new generations of superior machines after the shock. It is also clear from Figure 2 that maintenance of machines of certain vintage decreases with the age (cf. Lemma 2). It is remarkable that the machines of the superior technology are maintained more shortly after the technological shock than later. In fact, as also the investments shortly after $\bar{t}$ overshot the long run equilibrium corresponding to f_2, thus investment and maintenance appear once more as complements.

Remark 3 Clearly, $\bar{f}(t) \geq f(t)$ for every t. However, as the above numerical results show (as it is claimed also by Proposition 9 for the investments) that $\bar{I}(t) < I(t)$ and $\bar{m}(t,a) < m(t,a)$ for some t and a. This implies that the assumption of linearity of the maintenance cost is essential for the claim of Proposition 3.

6 Investments in old technologies

In the previous considerations we assumed that the firm can invest only in the newest currently available technology. In this section we allow for investments in old machines, denoted by $J(t,a)$, where a indicates the age of the technology. The possibility to invest in old machines leads to the following changes in the basic model (1)–(6). The equation of the dynamics of the capital becomes

$$K_t + K_a = -\delta(a, m(t,a))K(t,a) + J(t,a), \quad K(0,a) = K_0(a), \quad K(t,0) = I(t).$$

The objective function includes the cost of investment in old machines:

$$\max \int_0^T e^{-rt} \left[pQ(t) - C(I(t)) - D(M(t)) - \int_0^\omega C^{\text{old}}(a, J(t,a))\, \mathrm{d}a \right] \mathrm{d}t,$$

where $C^{\text{old}}(a,J)$ is the cost of installation of J machines of age a.

The adjoint equation (8) remains the same as in the case of investments in new machines only. The maximal principle from [19] implies that

$$J(t,a) = ((C^{\text{old}})')^{-1}(\xi(t,a)), \tag{21}$$

where now $J(t,a)$ is allowed to be also negative. Therefore a state constraint $K(t,a) \geq 0$ should be added to the model, in principle, but in the numerical results below it is automatically fulfilled.

A remarkable feauture of the analysis is that propositions 3–5 are still valid also in the case of investment in old machines.

Below we investigate numerically (see footnote 4) what is the impact of the possibility to invest in old machines on the optimal investment and maintenance. We consider a firm with the same parameters as in Section 5 with the additional specification

$$C^{\text{old}}(a,J) = b^{\text{old}}(a)J + 0.5c^{\text{old}}J^2, \quad \text{with } b^{\text{old}}(a) = b * (1 - a/\omega), \;\; c^{\text{old}} = c.$$

Figures 3, 4, and 5 represent the optimal investments in new machines, the optimal investment in old machines, and the optimal maintenance, respectively. Comparing with figures 1 and 2 we see that the investment in old machines appears as a substitute of both investments in new machines and maintenance. It is remarkable that the relative drop of investment in new machines before the technological shock at $\bar{t} = 30$ is larger if investments in old machines are allowed. The reason is, that a part of the investments in new machines shortly before the technological breakthrough at time $\bar{t}$ (see Figure 1) are replaced by investments in old machines, which will be scrapped shortly after $\bar{t}$ so that they also can be replaced by the superior machines. This leads to a second anticipation effect, namely, that there will be more scrapping due to maximal age, ω, soon after $\bar{t}$, compared with the case of $J(t,a) = 0$, hence the still existing machines will be maintained more. This is clearly seen on Figure 5: the peak of $m(t,a)$ for $\bar{t} < t - a < \bar{t} + 3$ is much higher than the one on Figure 2, both in relative and in absolute terms.

[Figure 3 about here.]

[Figure 4 about here.]

[Figure 5 about here.]

7 Appendix

Proof of Proposition 1. (A sketch.) First of al, it is not difficult to argue that any minimizing sequence $\{(I_k, m_k)\}$ is bounded in L_∞: for I, due to the strong convexity of

C, and for m – due to the finite life time of the physical capital and $D'(M) \geq \gamma^D > 0$. Then the main argument is standard, in principle: the L_2-weak lower semi-continuity of integral functionals with convex integrands.

The function $C(I)$ is supposed convex and I enters linearly in the (boundary condition of) differential equation (2). The function $D(m)$ is also supposed convex, but the main trouble is caused by the nonlinearity of (2) with respect to m. However, the solution of equation (2) has an explicit (Cauchy-type, along the characteristics) representation, where δ is comfortably integrated. This makes it possible to use the convexity of δ to prove that the weak limit of $\{m_k\}$ gives an objective value that is not worse than the limit of the objective values provided by m_k. We skip the details of this proof. Q.E.D.

Proof of Lemma 1. On every characteristic line starting at a point $(t, 0)$ (that is on the segment $\{(t+s, s) : s \in [0, \omega]\}$) the solution ξ of the adjoint equation (8) satisfies an ordinary differential equation. Namely, $\eta(s) = \xi(t+s, s)$ satisfies

$$\eta'(s) = (r + \delta(s, m(t+s, s)))\eta(s) + pf(t) - D'(M(t+s))(m_0 + m(t+s, s)), \qquad (22)$$

with $\eta(\omega) = 0$. For $\eta = 0$ the right-hand side is

$$\eta' = pf(t) - D'(M(t+s))(m_0 + m(t+s, s)).$$

If it happens that $K(t+s, s) = 0$, then $m(t+s, s) = 0$ due to the convention made at the end of Section 2. Otherwise, we also have $m(t+s, s) = 0$ due to (10) and the assumption that $D'(M(t+s)) \geq \gamma^D$. Thus we have

$$\eta' = pf(t) - D'(M(t+s))m_0 \leq pf(t+s) - D'(M(t+s))m_0 < 0,$$

due to the monotonicity of f and the regularity assumption. Thus the region $\eta \geq 0$ is invariant with respect to (22) backward in time, and its boundary strictly repels the trajectories. Q.E.D.

Proof of Proposition 5. We employ again a comparison argument for the solutions η_1 and η_2 of equation (16), corresponding to the two functions δ_1 and δ_2. For any fixed $t \in [0, T-\omega]$ we have to prove that $\eta_1(s) \geq \eta_2(s)$ for every $s \in [0, \omega]$. Then the claim of the proposition would follow from Remark 1. That is, we have to prove that if $\eta_1 = \eta_2 = \eta > 0$ at some point s, then $\eta_1' \leq \eta_2'$, which is equivalent to

$$\delta_1(s, \hat{m}_1(s, \eta))\eta + d\hat{m}_1(s, \eta) \leq \delta_2(s, \hat{m}_2(s, \eta))\eta + d\hat{m}_2(s, \eta),$$

where

$$\hat{m}_i(s,\eta)) = (\delta'_{i,m})_+^{-1}\left(s, -\frac{d}{\eta}\right).$$

Substituting $y = -d/\eta$, we see that we have to prove that

$$\delta_1(s,(\delta'_{1,m})_+^{-1}(s,y)) - (\delta'_{1,m})_+^{-1}(s,y)y \le \delta_2(s,(\delta'_{2,m})_+^{-1}(s,y)) - (\delta'_{2,m})_+^{-1}(s,y)y \tag{23}$$

for every $y < 0$. To do this we consider three cases.

1. Let $y \le \delta'_{1,m}(s,0)$. Then, according to condition (ii) also $y \le \delta'_{2,m}(s,0)$. Then

$$(\delta'_{1,m})_+^{-1}(s,y) = (\delta'_{2,m})_+^{-1}(s,y) = 0,$$

and (23) is implied by $\delta_1(s,0) \le \delta_2(s,0)$.

2. Let $\delta'_{1,m}(s,0) < y < \delta'_{2,m}(s,0)$. Then substituting $m_i(s,y) = (\delta'_{i,m})_+^{-1}(s,y)$ and taking into account that $y = \delta'_{1,m}(s,m_1(s,y)))$ we have

$$\begin{aligned}\delta_1(s,(\delta'_{1,m})_+^{-1}(s,y)) - (\delta'_{1,m})_+^{-1}(s,y)y &= \delta_1(s,(\delta'_{1,m})^{-1}(s,y)) - (\delta'_{1,m})^{-1}(s,y)y \\ &= \delta_1(s,m_1(s,y)) - \delta'_{1,m}(s,m_1(s,y)))m_1(s,y) \le \delta_1(s,0) \\ &\le \delta_2(s,0) = \delta_2(s,(\delta'_{2,m})_+^{-1}(s,y)) - (\delta'_{2,m})_+^{-1}(s,y)y,\end{aligned}$$

where the inequality in the second last line follows from the convexity of δ_1 with respect to m. This verifies (23) also in this case.

3. Let $\delta'_{1,m}(s,0) \le \delta'_{2,m}(s,0) < y$. In this case inequality (23) can be equivalently rewritten without the "+" subscripts. Using again the notation $m_i(s,y)$ (where now the "+" subscript is omitted), we represent (23) in the following way:

$$\delta_1(s,m_1(s,y)) - ym_1(s,y) \le \delta_2(s,m_2(s,y)) - ym_2(s,y),$$

where $y = \delta'_{1,m}(s,m_1(s,y))) = \delta'_{2,m}(s,m_2(s,y)))$. Then the proposition follows from the following claim.

Claim. Let for two differentiable convex functions g_1 and g_2, defined on $[0,\infty)$, it hold that $g_1(m) \le g_2(m)$ for every $m \ge 0$. Let m_1 and m_2 be two points such that $g'_1(m_1) = g'_2(m_2)$. Then

$$g_1(m_1) - g'_1(m_1)m_1 \le g_2(m_2) - g'_2(m_2)m_2.$$

The proof is straightforward:

$$g_2(m_2) \ge g_1(m_2) \ge g_1(m_1) + g'_1(m_1)(m_2 - m_1)$$

$$= g_1(m_1) + g_1'(m_1)m_2 - g_1'(m_1)m_1 = g_1(m_1) + g_2'(m_2)m_2 - g_1'(m_1)m_1,$$

where the first inequality follows from the convexity of g_1. This implies the claim, hence, completes the first statement of the proposition.

To prove the second statement of the proposition one should just reconsider the above inequalities having in mind the additional assumptions. Q.E.D.

Proof of Lemma 2. Since δ_m is independent of a, $\hat{m}$ is also independent of a, therefore below we skip the argument a of $\hat{m}$.

If $m(t,a) > 0$ we have

$$m_a(t,a) = \hat{m}_\xi(\xi(t,a))\xi_a(t,a).$$

Having in mind the definition of the function $\hat{m}$ in (13) and applying standard calculus one obtains that

$$\hat{m}_\xi(\xi(t,a)) = [\delta''_{mm}(\hat{m}(\xi))]^{-1} \frac{d}{\xi^2} > 0,$$

where the inequality is due to the standing assumption (ii). Differentiating the adjoint equation (15) in a we obtain for $\nu(t,a) = \xi_a(t,a)$ the equation

$$\nu_t + \nu_a = [...]\nu(t,a) + \delta_a(a, \hat{m}(\xi(t,a)))\xi(t,a) + pf'(t-a), \quad \nu(t,\omega) \le 0,$$

where the term in the brackets is of no importance, and the inequality $\nu(t,\omega) \le 0$ follows from $\xi(t,\omega) = 0$, $\xi(t,a) \ge 0$. Since $f' \ge 0$, $\delta_a \ge 0$, and $\xi(t,a) \ge 0$ (according to Lemma 1), the solution $\nu(t,a)$ is non-positive, by the same argument used in the proof of Proposition 3. Q.E.D.

Proof of Proposition 8. We remind that we consider the control variables I and m as elements of the spaces $L_\infty([0,T])$ and $L_\infty([0,T] \times [0,\omega])$, respectively. The proof of the maximum principle in [19] (see also Proposition 1 in [41]) implies that the functional $m \longrightarrow V(\tilde{I}, m)$ is Gâteaux differentiable and its derivative $d_m V$ has a functional representation, namely (compare with the derivative with respect to m of the expression in the right-hand side of (10))

$$\mathrm{d}_m V(\tilde{I}, m)(t,a) = [\tilde{\xi}(t,a)\delta'_m(a, m(t,a)) - D'(\tilde{M}(t))]\tilde{K}(t,a),$$

where $\tilde{K}$ and $\tilde{M}$ correspond to the control pair $(\tilde{I}, m)$, and $\tilde{\xi}$ is the corresponding solution of the adjoint equation (8). Clearly, to prove the proposition it is enough to show that

$\mathrm{d}_m V(\tilde{I}, m)(t,a) \geq 0$, for all $(t,a) \in [0, T-\omega] \times [0,\omega)$ and the inequality is strict whenever $m(t,a) > 0$.

Due to the assumption that $I(t) > 0$ for all $t < T$, for $\tilde{I}$ as in Definition 1 we have $\tilde{I}(t) > 0$, hence $\tilde{K}(t,a) > 0$ for all $t \in [\omega, T)$ and $a \in (0,\omega]$. Then it remains to prove that

$$\tilde{\xi}(t,a)\delta'_m(a, m(t,a)) - D'(\tilde{M}(t)) \geq 0, \quad \text{for } (t,a) \in [0, T-\omega] \times [0,\omega)$$

and the inequality is strict whenever $m(t,a) > 0$.

According to the maximum principle, for (t,a) for which $m(t,a) > 0$ we have that

$$\xi(t,a)\delta'_m(a, m(t,a)) - D'(M(t)) = 0,$$

hence it suffices to prove the follow two inequalities:

$$\tilde{\xi}(t,a) \geq \xi(t,a), \quad D'(\tilde{M}(t)) \leq D'(M(t)), \tag{24}$$

where the latter one is strict for those $t \in [\omega, T)$ for which $m(t,a) > 0$ on a set of positive measure. Clearly, for $t \in [\omega, T]$

$$\tilde{M}(t) = \int_0^\omega (m_0 + m(t,a))\tilde{K}(t,a)\,\mathrm{d}a < \int_0^\omega (m_0 + m(t,a))K(t,a)\,\mathrm{d}a = M(t),$$

since obviously $\tilde{K}(t,a) < K(t,a)$ for all $t \in [\omega, T)$ and $a \in (0,\omega]$. Then the condition for D implies $D'(M(t)) \geq D'(\tilde{M}(t)) + \rho(M(t) - \tilde{M}(t)) > D'(\tilde{M}(t))$. The first inequality in (24) follows from the comparison argument for the solutions of the adjoint equation used several times above. Here the equations for $\tilde{\xi}$ and ξ differ only in the term $D'(\tilde{M}(t)) \leq D'(M(t))$, which implies the desired inequality. Q.E.D.

Proof of Lemma 3. The proof does not use the particular forms of f given by (19) and (20), therefore we skip the bars in the notations $\bar{M}$, $\bar{Q}$, etc. First we remind that the function $(\delta'_m)_+^{-1}$ is Lipschitz continuous, and the function $\xi \longrightarrow (\delta'_m)_+^{-1}\left(a, -\frac{D}{\xi}\right)$ is also Lipschitz continuous in $[0, +\infty)$. Moreover, below we use that $(\delta'_m)_+^{-1}(a, y)$ is monotone increasing in its second argument, and also that the adjoint function ξ is Lipschitz continuous along every characteristic line $\{(t+a, a) : \ a \in [0,\omega]\}$.

Consider t', $t'' \in [0, T-\omega]$ and let $t'' = t' + \varepsilon$, where $|\varepsilon|$ is a sufficiently small number, so that the manipulations below make sense. Without any restriction we assume that

$$M(t') \leq M(t''). \tag{25}$$

Further, c_1, $c_2, \ldots$ denote constants that are independent of t' and t''. We have

$$\begin{aligned} m(t',a) &= (\delta'_m)^{-1}_+ \left(a, -\frac{D'(M(t'))}{\xi(t',a)}\right) \geq (\delta'_m)^{-1}_+ \left(a, -\frac{D'(M(t''))}{\xi(t',a)}\right) \\ &\geq (\delta'_m)^{-1}_+ \left(a, -\frac{D'(M(t''))}{\xi(t'',a+\varepsilon)}\right) - c_1|\varepsilon| \geq (\delta'_m)^{-1}_+ \left(a+\varepsilon, -\frac{D'(M(t''))}{\xi(t'',a+\varepsilon)}\right) - c_2|\varepsilon| \\ &= m(t'',a+\varepsilon) - c_2|\varepsilon|. \end{aligned}$$

Using this inequality and the Lipschitz continuity of K along every characteristic line $\{(t+a,a):\ a \in [0,\omega]\}$, we obtain that

$$\begin{aligned} M(t'') &= \int_0^\omega m_0 K(t'',a)\,\mathrm{d}a + \int_0^\omega m(t'',a)K(t'',a)\,\mathrm{d}a \\ &\leq \int_\varepsilon^{\omega-\varepsilon} m_0 K(t'',a)\,\mathrm{d}a + \int_\varepsilon^{\omega-\varepsilon} m(t'',a)K(t'',a)\,\mathrm{d}a + c_3|\varepsilon| \\ &= \int_\varepsilon^{\omega-\varepsilon} m_0 K(t'+\varepsilon,a)\,\mathrm{d}a + \int_0^{\omega-2\varepsilon} m(t'',a+\varepsilon)K(t'',a+\varepsilon)\,\mathrm{d}a + c_3|\varepsilon| \\ &\leq \int_\varepsilon^{\omega-\varepsilon} m_0 K(t',a-\varepsilon)\,\mathrm{d}a + \int_0^{\omega-2\varepsilon} (m(t',a)+c_2|\varepsilon|)K(t',a)\,\mathrm{d}a + c_4|\varepsilon| \\ &\leq \int_0^{\omega-2\varepsilon} m_0 K(t',a)\,\mathrm{d}a + \int_0^{\omega-2\varepsilon} m(t',a)K(t',a)\,\mathrm{d}a + c_5|\varepsilon| \\ &\leq M(t') + c_6|\varepsilon|. \end{aligned}$$

Taking into account (25) we obtain that

$$|M(t'') - M(t')| \leq c_6|t' - t''|,$$

which proves the first claim of the lemma.

To prove the Lipschitz continuity of Q it is enough to use again the Lipschitz continuity of K along the characteristic lines. Indeed, with the standard use of $O(h)$ we have for any sufficiently small h that

$$Q(t+h) = \int_0^\omega f(t+h-a)K(t+h,a)\,\mathrm{d}a = \int_0^\omega f(t-(a-h)))K(t,a-h)\,\mathrm{d}a + O(h)$$

$$\int_{-h}^{\omega-h} f(t-a))K(t,a)\,\mathrm{d}a + O(h) = \int_0^\omega f(t-a)K(t,a)\,\mathrm{d}a + O(h) = Q(t) + O(h).$$

This completes the proof of the lemma. Q.E.D.

It is useful to note that the Lipschitz constants are uniform in f when f is bounded.

Proof of Lemma 4. We consider again the function $\eta(s) = \xi(t+s, s)$ (where t is fixed), which satisfies equation (22) with $\eta(\omega) = 0$. ¿From the maximum principle we have that

$$m(t+s, s) = (\delta'_m)_+^{-1}\left(s, -\frac{D'(M(t+s))}{\eta(s)}\right) := \mu(t+s, s, \eta(s)).$$

Thus η satisfies also the equation

$$\eta'(s) = (r + \delta(s, \mu(t+s, s, \eta(s))))\eta(s) - pf(t) + D'(M(t+s))(m_0 + \mu(t+s, s, \eta(s))),$$

Now we fix two different values $t = t_1 < \bar{t}$ and $t = t_2 > \bar{t}$ and denote by η_1 and η_2 the corresponding solutions of the above equation. Notice that we have $f(t_1) = f_1$ and $f(t_2) = f_2$.

Now we shall estimate the difference $\dot{\eta}_2(s) - \dot{\eta}_1(s)$ for those s for which $\eta_2(s) \geq \eta_1(s)$. In the estimation we use that μ is Lipschitz and monotone increasing with respect to η (see end of Section 3), that M is Lipschitz (see Lemma 3), and that, as a consequence of the latter, μ is Lipschitz with respect to t. Then

$$\begin{aligned}
&\dot{\eta}_2(s) - \dot{\eta}_1(s) \\
&= [(r + \delta(s, \mu(t_2+s, s, \eta_2(s)))\eta_2(s) - (r + \delta(s, \mu(t_1+s, s, \eta_2(s)))\eta_2(s)] - p(f_2 - f_1) \\
&+[D'(M(t_2+s))(m_0 + \mu(t_2+s, s, \eta_2(s))) - D'(M(t_1+s))(m_0 + \mu(t_1+s, s, \eta_1(s)))] \\
&\leq [r + \delta(s, \mu(t_1+s, s, \eta_1(s)))](\eta_2 - \eta_1) \\
&+[\delta(s, \mu(t_1+s, s, \eta_2(s))) - \delta(s, \mu(t_1+s, s, \eta_1(s)))]\eta_2(s) + c_1(t_2 - t_1) - p(f_2 - f_1) \\
&+D'(M(t_1+s))[\mu(t_1+s, s, \eta_2(s))) - \mu(t_1+s, s, \eta_1(s)))] + c_2(t_2 - t_1),
\end{aligned}$$

where c_1, $c_2, \ldots$ are independent of t_1, t_2, f_1 and f_2 (for $f_1 \leq f_2$ faking values in a compact interval). Using that μ is increasing in η and δ is decreasing in m we observe that the expression in the second brackets is non-positive, while the one in the last brackets is non-negative. Then due to the monotonicity and Lipschitz continuity of μ

$$\dot{\eta}_2(s) - \dot{\eta}_1(s) \leq [r + \delta(s, \mu(t_1+s, s, \eta_1(s)))](\eta_2 - \eta_1) - p(f_2 - f_1) + c_4(\eta_2 - \eta_1) + c_3(t_2 - t_1).$$

Since δ is positive and bounded we obtain that for s for which $\eta_2(s) \geq \eta_1(s)$ it holds

$$\dot{\eta}_2(s) - \dot{\eta}_1(s) \leq c_5(\eta_2(s) - \eta_1(s)) - p(f_2 - f_1) + c_3(t_2 - t_1),$$

Since $\eta_1(\omega) = \eta_2(\omega) = 0$, and since $p(f_2 - f_1) > c_3(t_2 - t_1)$ for t_1 and t_2 sufficiently close to $\bar{t}$ we conclude that $\eta_2(s) > \eta_1(s)$ for every $s \in [0, \omega)$. Moreover, from the Cauchy formula,

$$\eta_2(s) - \eta_1(s) \geq [p(f_2 - f_1) - c_3(t_2 - t_1)] \int_s^\omega e^{-c_5(\theta - s)}\,\mathrm{d}s \geq c_6[p(f_2 - f_1) - c_3(t_2 - t_1)](\omega - s).$$

Then the claims of the lemma follow from the representations of the optimal controls by the formulas (12). Q.E.D.

Proof Proposition 9. Since we know that $\bar{I}$ and I are continuous in $(\bar{t} - 2\omega, \bar{t})$, it is enough to prove that $\bar{I}(t) < I(t)$ for some t. Assume that $\bar{I}(t) \geq I(t)$ for all t from this interval. Then (due to the assumption that $\bar{m} = m$) $\bar{K}(t + a, a) \geq K(t + a, a)$ for all $t \in (\bar{t} - \omega, \bar{t}]$ and $a \in [0, \omega]$. This implies

$$\bar{M}(t) \geq M(t) \quad \text{for } t \in [\bar{t} - \omega, \bar{t}],$$

and also

$$\bar{K}(t, a) \geq K(t, a) \quad \text{for } t \in [\bar{t}, \bar{t} + \omega], \;\; a \in [t - \bar{t}, \omega]. \tag{26}$$

On the other hand, according to lemmas 3 and 4 we have for for $t > \bar{t}$

$$\bar{I}(t) \geq \bar{I}(\bar{t} + 0) - c_1(t - \bar{t}) \geq I(\bar{t} - 0) + c_2(f_2 - f_1) - c_1(t - \bar{t})$$

$$\geq I(\bar{t} - 0) + c_2(f_2 - f_1) - c_1(t - \bar{t}) \geq I(t) + c_2(f_2 - f_1) - c_3(t - \bar{t}),$$

where $c_1, \; c_2, \ldots$ are appropriate constants. Thus for a $t^* > \bar{t}$ which is sufficiently close to $\bar{t}$ and for some $\gamma > 0$ it holds that

$$\bar{I}(t) \geq I(t) + \gamma(t - \bar{t}) \quad \text{for } \; t \in (\bar{t}, t^*).$$

As before, this implies

$$\bar{K}(t, a) \geq K(t, a) + c_4\gamma \quad \text{for } t \in [\bar{t}, t^*], \;\; a \in [0, t - \bar{t}].$$

Using this inequality and (26) we obtain that for $t \in [\bar{t}, t^*]$

$$\bar{M}(t) = \int_0^{t-\bar{t}} [m_0 + m(t, a)]\bar{K}(t, a) \, \mathrm{d}a + \int_{t-\bar{t}}^{\omega} [m_0 + m(t, a)]\bar{K}(t, a) \, \mathrm{d}a$$

$$\geq \int_0^{t-\bar{t}} [m_0 + m(t, a)][K(t, a) + c_4\gamma] \, \mathrm{d}a + \int_{t-\bar{t}}^{\omega} [m_0 + m(t, a)]K(t, a) \, \mathrm{d}a$$

$$M(t) + \int_0^{t-\bar{t}} [m_0 + m(t, a)]c_4\gamma \, \mathrm{d}a \geq M(t) + c_5(t - \bar{t}).$$

To complete the proof it is enough to consider equation (22) for the anticipating and for the non-anticipating firm for $t \in (\bar{t} - \omega, t^* - \omega)$. Since we obtained that $\bar{M}(t+s) \geq M(t+s)$

and the inequality is strict for $s > \bar{t} - t$, and since $f(t) = f_1$ in both cases, we conclude that that the solution $\bar{\eta}$ of the anticipating firm, and η of the non-anticipating one, satisfy

$$\bar{\eta}(0) < \eta(0),$$

which implies $\bar{I}(t) < I(t)$. The proof is complete. Q.E.D.

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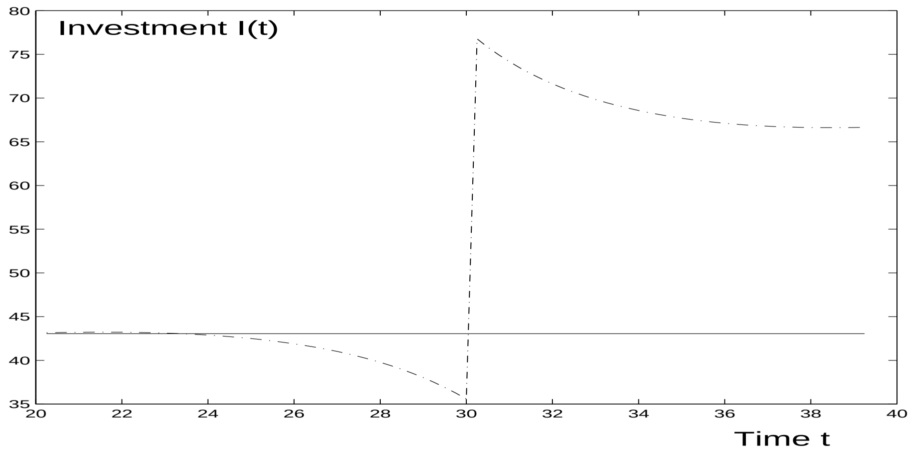

Figure 1: Optimal investment if a technological breakthrough at time $\bar{t} = 30$ is anticipated.

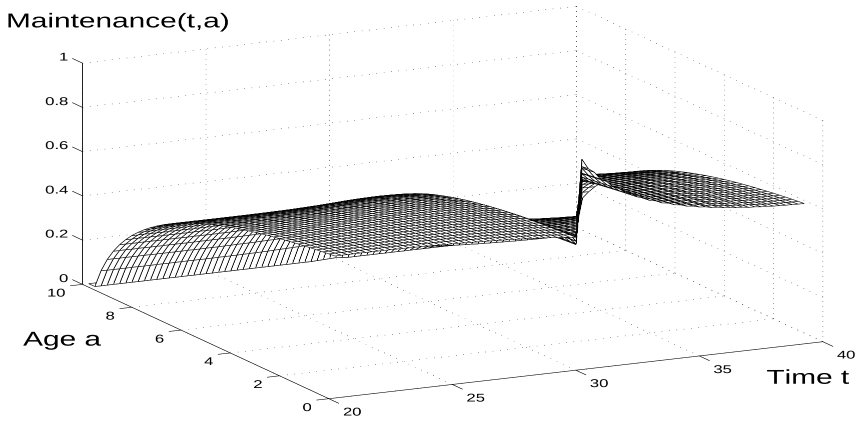

Figure 2: Optimal maintenance if a technological breakthrough at time $\bar{t} = 30$ is anticipated.

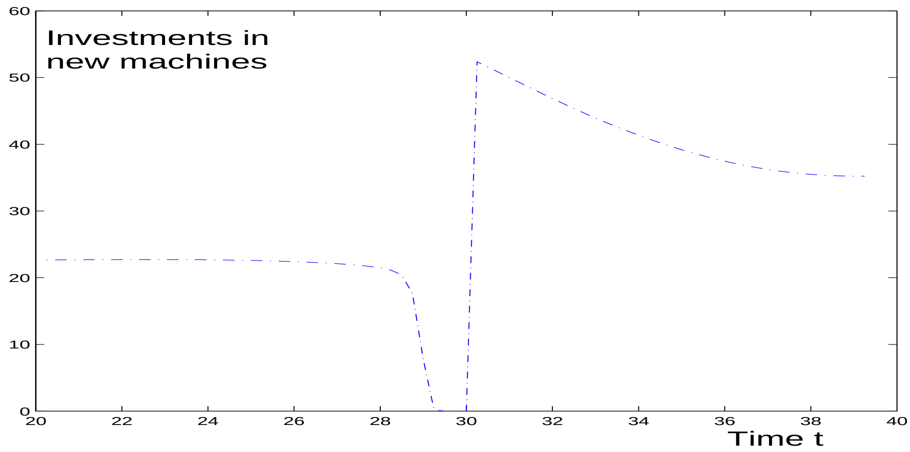

Figure 3: Optimal investment in new machines if a technological breakthrough at time $\bar{t} = 30$ is anticipated.

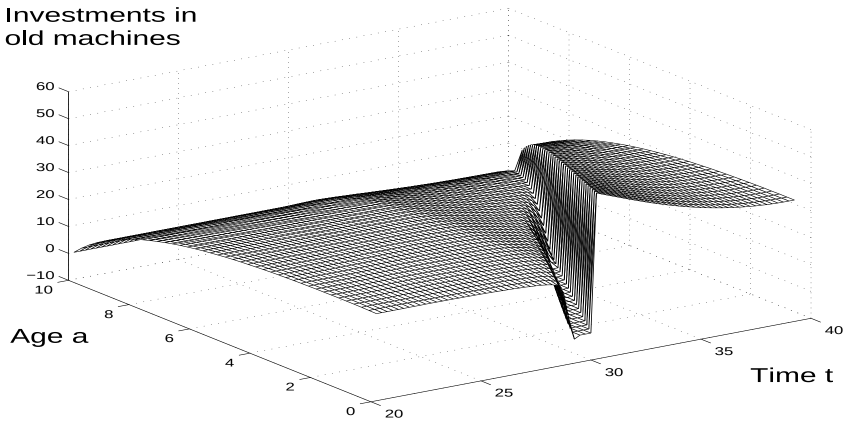

Figure 4: Optimal investment in old machines if a technological breakthrough at time $\bar{t} = 30$ is anticipated.

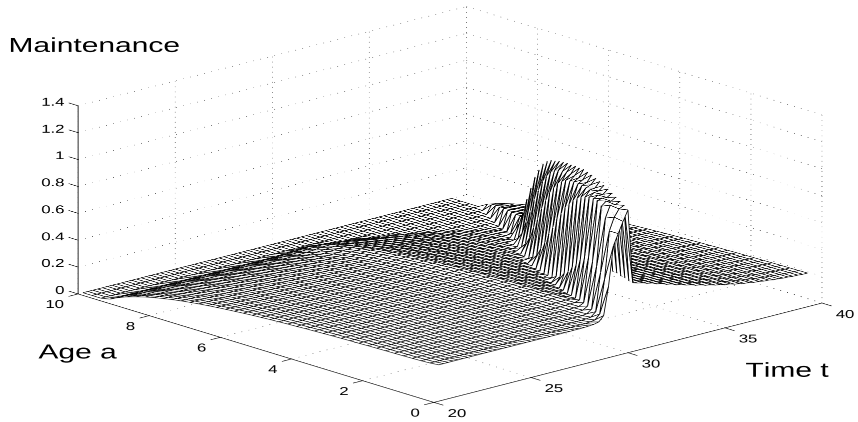

Figure 5: Optimal maintenance if investments in old machines are allowed and if a technological breakthrough at time $\bar{t} = 30$ is anticipated.