How large should you be in a market? Unprofitable arbitrage and liquidity effects in a money economy

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Abstract. Unlike any other good a medium of exchange allows to trade in all markets. This paper will attempt to highlight that this limits the possibility of inconsistency/dispersion of prices. We explore the money market game with multiple trading posts per commodity type to clarify and qualify the role of liquidity constraints underlined in Koutsougeras [2003, J. Math. Econ., 39, 401-420]. Contrary to a barter economy [1990, J. Econ. Theory, 51,1,126-143], liquidity constraints are not critical for the violation of the law of one price. The failure of the law of one price hinges on the fact that some agents are very large in the market. Subsequently, we examine convergence to price uniformity as the number of agents grows, so as to stress that this convergence is linked to the agents' level of activity and not to liquidity effects.

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1 Introduction

What makes difference between barter and money economies is a very tricky question, which has occupied economists for years. The main difficulty has been that money is worthless in the Arrow-Debreu General Equilibrium model. This is why monetary theory has focused on the reasons of why a valueless medium of exchange can be accepted in trade. In particular, money facilitates trade. More precisely, it has been argued that the value of money is to circumvent the absence of double coincidence of wants in the exchange process (Ostroy and Starr, 1974, 1990). The use of money as a medium of exchange is indeed endogenously determined within the search model literature initiated by Kiyotaki and Wright (1989, 1991, 1993). In more recent analyses, money has been valued in models with double coincidence of wants which features asymetry (and inefficiency) in the bargains (Engineer and Shi (2001), Berentsen and Rocheteau (2003)). In such cases fiat money may be welfare improving. Concretely, money solves the asymmetric demand problem and enlarges the gains from trade. Ritter (1995) argues about the transition from barter to fiat money, in a model that follows in the vein of Kiyotacki-Wright. He combines two different features—the exchange of a valueless pieces of paper and the long time required for fiat money to become prevalent—to point out the role of a self-interest government in explaining the juxtaposition of both phenomena. The transition from barter to fiat money relies on the credibility of promises of the government to restrict the issue of money.

Another research program which has been followed by the strategic market game literature is to question "...what strategic limitations does the use of money impose on trade, what additional possibilities does it open up, and what are the institutional implications?" (Shapley and Shubik, 1977). In part, what is informative can be found in the conditions of equivalence between non-cooperative equilibrium of the market game and competitive equilibrium of the economy. In summary, this correspondence requires (a) that there is *enough money* in the economy—in the sense defined by Shubik [1990]), (b) that agents do not go bankrupt in equilibrium—which relies on the existence of efficient money institutions, and (c) that agents' influence on prices vanishes. [2] More recently, in a different approach that follows in the vein of the one period market game in which money is valued by its role as a medium of

[2] One can refer to Shapley and Shubik (1977), Postlewaite and Schmeidler (1978), Shubik and Wilson (1977), Dubey and Shubik (1979), Shubik and Zhao (1991), Shubik and Tsomocos (1992), Dubey and Shapley (1994) and others, and for a more general perspective to Shubik (1990).

exchange (Dubey and Geanakoplos, 1992), Dubey and Geanakoplos (2003) show the central role of money in a finite multiperiod economy when there are missing assets and missing markets. The value of money is linked to its role in transactions when there are enough missing market links and if the inside-outside money ratio is large enough. In addition, this contribution is significant because the presence of inside and outside money circumvent problems which prevent existence of an equilibrium within the general equilibrium model with incomplete markets.

This paper follows this last approach to suggest another specificity of money economy that also hinges on the fact that money gives access to all markets. In short, we argue that money facilitates the uniformity of prices across markets. That is, the price inconsistency is more likely to occur in a barter economy than the violation of the law of one price in the money economy. Our explanation is the following. In absence of a generalized medium of exchange, an agent that is not endowed with some commodities can be excluded from the markets where these commodities are exchanged for other commodities. The consequence is that those agents may unfortunately induce or increase the dispersion of prices by trading on given trading-posts. Nonetheless, this cannot happen in the money economy because unlike other commodities money allows trade on all markets. The paper is precisely to argue for this point.

To present our contribution more precisely, let us introduce the background of these issues. From a more general perspective, it is generally expected that, in costless many markets economies, identical commodities exchange at the same price. In other words, dispersed prices are supposed to be an arbitrage opportunity which agents should immediately take advantage of, so that it would necessarily vanish. Yet, this generally accepted idea has been put into question within the market game framework which features more than one trading-post for each commodity type. In a market game with all pairwise markets available Amir and alii. (1990) show that the set of prices need not be consistent. Bloch and Ferrer (2001) and Koutsougeras (2003)——respectively in a bilateral oligopoly and in a (single) money market game with multiple trading-posts for each commodity type—show the existence of equilibrium featuring dispersion of prices. First and foremost, this calls for several observations. Firstly, in spite of how important and unexpected these results are, it would be dangerous to claim too much from the analysis of simple examples. [3] Then, it turns out that no fully-fledged explanation has been developed for

[3] For example Koutsougeras (2003*b*) claims "We can conclude that in a frictionless context, the lack of price taking is the only source of equilibria with arbitrage" (page 403). See Gobillard (2006) and Breton and Gobillard (2006) for a discussion.

the money market game so far. Among the different (partial) explanations that have been put forward, stress has particularly been put on the absence of the price taking assumption (Koutsougeras, 2003 and 2003*b*). It has then been highlighted by a convergence result as the number of agents grows (Koutsougeras, 2003*b*). Koutsougeras and Papadopoulos (2004) deal with strategic security markets and show the no-arbitrage principle does not survive imperfect competitive framework. They express equilibrium situations as elasticities of demand functions which are analogue to their expression in the competitive situation, in order to show that the competitive situation can only be achieved in the limit when the influence on prices vanishes. Nevertheless, we observe that these studies simply yield the conclusion that the price-taking assumption needs to be given up to defy the law of one price. That is, nothing can really be said about whether the incentives agents have to buy (sell) on the most (less) expensive posts. Moreover, the violation of the law of one price cannot be completely refutated by imperfect competition. Economic theory is full of examples that combine (costless) imperfect competitive situations and uniformity of prices.

Other interpretations have in turn been suggested, which, among others, are the market structure (Koutsougeras, 2003, 2003*b*) and more importantly the role of (commodity) liquidity constraints. In the commodity market game, price inconsistency is obtained when commodities are distributed in a skewed manner (Amir and alii., 1990). In this case, agents may be subject to liquidity constraints that limit their ability to benefit from arbitrage opportunities. This source of dispersion is taken up within the money market game by Koutsougeras (2003*b*). Assuming agents do not face liquidity constraints, as far as the number of agents grows the price dispersion vanishes, while when the number of unconstrained agents is limited the one price law is not ensured to be satisfied in the limit. On another side, the failure of the one price law relies on agents making the specific wash-sales transactions that cancel one another on a post (Gobillard, 2006) [4]. More importantly, the role of liquidity constraints is unclear because (money) liquidity constraints per se do not induce price dispersion (Breton and Gobillard, 2006).

Therefore, there are still several lessons to be learned from the analysis of the market game with multiple trading-posts per commodity. Here, particular attention will be paid to the *very* role of liquidity constraints. It is shown that liquidity constraints *inducing* price dispersion is the monopoly of the

[4] See also Bloch and Ferrer (2001) in the bilateral oligopoly case without liquidity constraints.

barter economy. To develop this idea more precisely the nature of liquidity constraints' effects has to be reconsidered. The arguments called on until then in the literature is liquidity effects to be an obstacle to arbitrage strategies. Nonetheless, their main impact is to be the origin of a price dispersion. Here is the proposal that can be made: *Liquidity constraints can induce price dispersion in the barter market game*. When commodities exchange for commodities, due to endowment constraints binding an agent who is not correctly furnished may be unable to exchange a given commodity across all the trading posts where this commodity is exchanged. In this case, he is not able to enter (first) the posts with the most interesting prices and so induces or increases the price dispersion by trading on other trading-posts.

As mentioned before, it is the line of argument of the paper to show this cannot happen in the money market game. The key to the difference in these results lies in the fact that the structure of markets and trade are not similar: money allows to trade on all markets and to enter markets with interesting prices first. Hence, liquidity constraints may at most limit the ability to arbitrage a dispersion of prices in the money market game and never induce or increase price dispersion. Two lines of arguments are followed to understand this idea. We start with an intuitive result that anticipates the others. Given moves of others, a comparison of the price dispersion before and after a given agent plays his best reply states that any agent always reduces the dispersion of prices. Therefore, one can conclude first that liquidity constraints can only be an obstacle to this feature. A second and more detailed analysis comes next. Observe first an offer strategy can increase or induce price dispersion only if the offer strategies on less expensive posts q_h^{le} are greater than the one q_h^{me} on the most expensive one. We therefore express q_h^{le} as a function of q_h^{me} and show q_h^{le} is a zero function until q_h^{me} exceeds a critical value and becomes an increasing function of q_h^{me} next. However, the gradient is smaller than one so that price dispersion is a decreasing function of q_h^{le}, and the effect of liquidity constraints follows. As a result, the impact of endowment constraints is weaker in the money market game and price dispersion is more likely to occur in the barter economy.

We then turn to the issues of convergence to the law of one price when the number of agents grows. Recall Koutsougeras (2003*b*) underlines the importance of the number of unconstrained agents. Nevertheless, the fact that agents are not constrained is not necessary to the convergence to price uniformity. In this respect, we show that the dispersion vanishes as the number of agents trading each commodity type increases, whether there are constrained or not. This last condition can indeed be obtained when the number of agents endowed with a given commodity increases. When prices for a commodity are not uniform all agents endowed with this commodity exchange it, so that our convergence theorem applies to situations when the number of endowed agents increases. Here is the connection with the result of Koutsougeras, as agents whose endowment is zero are supposed to be constrained. It is actually not surprising that convergence results shed some light on the other features of the paper, that is that dispersion/uniformity of prices is linked to agents' levels of activity more than to the number of unconstrained agents.

This set of results suggests the following observation. Liquidity constraints do not yield sufficient conditions to obtain a dispersion of prices in the money market game. If exogenous constraints cannot explain price dispersion, it is expected that price dispersion is due to arbitrage opportunities that turn out to be non-profitable, that is, agents that do not *want* to *fully* arbitrage a price difference. Therefore, this strangeness of economic behaviors clearly asks for a deeper explanation, which is our second concern in this paper. In Gobillard (2006) it is shown that a seller on a cheap trading-post makes a larger volume of trade on more expensive trading-posts. Here, we want to reverse the question and identify the *incentives* agents have to *enter* trading-posts with less interesting prices. It is highlighted that what appears to be crucial for equilibrium price dispersion is the relative level of activity agents have on different trading-posts. What is meant here is that price dispersion hinges on some agents becoming (relatively) large on the market. We proceed by studying arbitrage behaviors when prices diverge. We use the market game in which agents are not allowed to enter the different posts on both sides as a tool, so as to express arbitrage behaviors in a simpler way. Recall price dispersion occurs when wash-sales are traded only. Nevertheless, this (out of equilibrium) analysis is full of lessons to be learned because there are arbitrage opportunities from dispersed prices that turn out to be non-profitable. As mentioned before, we show this situation occurs when some agents becomes large on a market.

The key point for this feature is that a marginal move does affect the entire set of allocations gotten on the post. In other words, any action impacts negatively on the transaction it aims at achieving. In consequence, when the relative weight of the agent exceeds a limit proportion, this negative *marginal unit* effect becomes so large that the agent is better off if he enters another trading-post even if its price is less interesting. The reasons for the failure of the law of one price can therefore be identified. In few words, wash-sales traded on a post by several agents increase its thickness and diminish the negative influence others may have on prices. Hence, conditions that rule out profitable arbitrage opportunities are less strict when wash-sales are traded on the posts with less interesting prices and equilibrium prices featuring dispersion is more likely to occur.

In a last section, an assessment of the importance of the market structure is developed through a comparison of the equilibrium allocation sets when the number of trading-posts changes. More precisely, some results of Koutsougeras (2003) are extended to situations with liquidity considerations. [5] To sum up, increasing the number of trading-posts only adds equilibrium allocations that contravene the one law of one price and above all contain wash-sales. By way of conclusion, the interest of working with the market game with multiple trading-posts instead of the canonical variant with one post per commodity type is quite limited. [6]

The paper is organized as follows. The framework is presented in section 2. Section 3 analysis a simple example and the general setting is investigated in section 4. Results about price dispersion and liquidity effects are stated in section 5, and the convergence to price uniformity in section 6. The influence of the market structure is analyzed in section 7 and the last section concludes the paper.

2 The framework

Throughout this study it will be used the market game with multiple trading posts per commodity type built by Koutsougeras (2003). It extends the Bid and Offer market game without liquidity constraint [7] (Postlewaite and Schmeidler, 1978) which analysis has been improved by Peck, Shell and Spear (1992).

There are a finite set H of N agents h and L commodity types $i = 1, ..., L$ in the economy. Each agent $h \in H$ is characterized by his preferences represented by a utility function $u_h : I\!R^L_+ \rightarrow I\!R$ which is assumed to be strictly concave and smooth increasing over x_h, and an initial endowment $e_h = \left\{ e^1_h, ..., e^L_h \right\} \in$

[5] It is noteworthy that focusing on free liquidity constraints situations is quite restrictive in this setup (see below). Bloch and Ferrer (2001) obtain similar results as to Koutsougeras (2003*b*).

[6] For a complete analysis of this feature which hinges on the application of a robustness requirment, we refer to Breton and Gobillard (2006).

[7] Definitions come next.

$I\!R^L$ where e_h^i is h's endowment of commodity i.

The economy is organized into trading-posts where a commodity i is exchanged for units of account. K^i denotes the number of trading-posts for commodity i and (i, s) denotes the post $s = 1, ..., K^i$ at which this commodity is exchanged. On each trading-post (i, s) any agent h bids a non-negative quantity $b_h^{i,s}$ of unit of accounts and offers a non-negative quantity $q_h^{i,s}$ of commodity i. So, the strategy of an agent is a vector σ_h:

$$\sigma_h = \left\{ b_h^{i,s}, q_h^{i,s} : s = 1, .., K_i, \ i = 1, .., L \right\} \in \prod_{i=1}^{L} I\!R_+^{2K_i}$$

Agents can obtain as much units of accounts as they need but cannot sell more than their initial endowment, and the strategy set of each agent $h \in H$ is:

$$S_h = \left\{ \sigma_h \in \prod_{i=1}^{L} I\!R_+^{2K_i} : \sum_{s=1}^{K_i} q_h^{i,s} \leq e_h^i, i = 1, 2, ..., L \right\}$$

Hereafter, we will denote:

$$S = S_1 \times ... \times S_h \times ... \times S_{\#H}$$
$$S_{-h} = S_1 \times ... \times S_{h-1} \times S_{h+1} \times ... \times S_{\#H}$$

and $\sigma \in S$ a profile of strategies. Given $\sigma \in S$, let:

$$B^{i,s} = \sum_{h \in H} b_h^{i,s} \text{ and } Q^{i,s} = \sum_{h \in H} q_h^{i,s}$$

and, for each $h \in H$:

$$B_h^{i,s} = \sum_{n \in h, n \neq h} b_n^{i,s} \text{ and } Q_h^{i,s} = \sum_{n \in h, n \neq h} q_n^{i,s}$$

On each trading post (i, s), the price $p^{i,s}$ is defined so that the transactions clear according to the price formation rule:

$$p^{i,s} = \begin{cases} B^{i,s}/Q^{i,s} & \text{if } Q^{i,s} \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

and we denote

$$p^i = \left\{ p^{i,s} : p^{i,s} > 0, \ s = 1, ..., K^i \right\}$$

Finally, for each commodity l and each agent h, the final allocations are determined as follows:

$$x_h^i = \begin{cases} e_h^i - \sum\limits_{s=1}^{K_i} q_h^{i,s} + \sum\limits_{s=1}^{K_i} b_h^{i,s}/p^{i,s} & \text{if (3) holds} \\ e_h^i - \sum\limits_{s=1}^{K_i} q_h^{i,s} & \text{otherwise} \end{cases} \tag{2}$$

where it is postulated that $1/p^{i,s} = 0$ whenever $p^{i,s} = 0$. Specification (2) states that any agent h whose sales do not cover the value of his purchases has his purchases confiscated. This assumption ensures that agents do not go bankrupt in equilibrium, so that each consumer is considered as maximizing his utility function over strategies $\sigma_h \in S_h$ s.t. (3) holds.

$$\sum_{i=1}^{L}\sum_{s=1}^{K_i} b_h^{i,s} - \sum_{i=1}^{L}\sum_{s=1}^{K_i} q_h^{i,s} p^{i,s} \leq 0 \tag{3}$$

We will refer to this model as the Bid - Offer market game. It implicitly allows each agent h to enter a post (i,s) on both sides, $b_h^{i,s} > 0$ and $q_h^{i,s} > 0$. In that case, agent h is simultaneously buying and selling on the trading-post (i,s) and makes sales and purchases that cancel one another, namely, wash-sales.

Consider now the same model in which wash-sales are precluded. Formally, the model is the one exposed above, with each agent $h \in H$ facing an additional constraint on each trading-post:

$$b_h^{i,s} q_h^{i,s} = 0,\ i = 1, ..., L,\ \ s = 1, ..., K_i \tag{4}$$

In that case it will be referred to the Bid or Sell market game. Now, some notations and definitions are introduced.

Definition 1 *An active post (i,s) is defined as a trading-post where goods are exchanged at a strictly positive price, equivalently $B^{i,s} > 0$ and $Q^{i,s} > 0$, so that p^i is the set of prices on active post where commodity i is exchanged.*

Moreover, it will be said that, on a post (i,s), moves of other agents than agent h are strictly positive if $Q_h^{i,s} B_h^{i,s} > 0$.

Definition 2 *Let (p,x) denote a set of prices and final allocations achieved by a set of strategy profiles $\sigma \in S$ given (1) and (2), with $p \in I\!R_+^{K^1} \times ... \times I\!R_+^{K^L}$ and $x = \Big\{x_h \in I\!R_+^L : h \in H\Big\}$.*

Let us now formally define the law of one price. It is necessary to exclude prices on inactive posts for the law of one price to be defined as identical commodities exchanged at the same price. Actually, the zero prices posted on inactive posts violate the equality of prices. But, no commodity is exchanged at that price on these posts.

Definition 3 *A set of price and allocation (p, x) satisfies the law of one price if $\forall\ i = 1, ..., L$ prices across active trading-posts for commodity i are uniform:*

$$p^{i,s} = p^{i,r} : \ \forall\ p^{i,s}, p^{i,r} \in p^i$$

In the rest of the paper an equilibrium is defined to be a non-cooperative equilibrium.

Definition 4 *An equilibrium is a strategy set $\hat{\sigma} \in S$ such that for each $h \in H$:*

$$u\left(\hat{\sigma}_h, \hat{\sigma}_{-h}\right) = \sup_{\sigma_h \in S_h} u\left(\sigma_h, \hat{\sigma}_{-h}\right)$$

In this paper, we include situations when agents face commodity liquidity constraint. Following Koutsougeras [2003] an agent is said to be liquidity constrained if at least one of his i's endowment constraint is binding, implying the set of agents facing a liquidity constraint on market i is:

$$C_i\left(b, q\right) = \left\{ h \in H : \sum_{i=1}^{K^i} q_h^{i,s} = e^i ; b^{i,s} q^{i,s} = 0 \right\} \tag{5}$$

The no wash-sales condition is to exclude situations when an agent offers all of his commodity endowment but is not really constrained as he is trading wash-sales. Now, according to definition (5) focusing on situations where $C_i\left(b, q\right) = \emptyset$ may be very restricting, as no agent is supposed to be constrained and all agents are supposed to be endowed of each commodity. Moreover, endowment constraints are more likely to be binding if agents have the possibility to sell their entire endowment and to buy back the same good simultaneously.

3 A simple example

This section explores a simple (out of equilibrium) example. Its purpose is to make clear the underlying mechanisms at play and to introduce to the equilibrium characteristics. This heuristic analysis may help to explain why the wash-sales and volumes of trade matter, and why liquidity constraints do not. Our strategy is to look at how an agent h transferring a unit of account from the expensive post to a cheapest one affects his situation. There are one commodity and two posts $s = 1$ and $r = 2$. b^s and q^s are the quantities of money and goods that agent h puts on each post $s = 1, 2$. B^s and Q^s with $s = 1, 2$ are the quantities of money and good deposited on each post and p^1 and p^2 are the prices.

3.1 Intuitions from a picturesque situation

We start with a peculiar situation, where $Q^1 = Q^2 = 1$, $B^1 = 6$ and $B^2 = 4$, so that prices are $p^1 = 6$ and $p^2 = 4$.

The marginal unit effect. Assume agent h plays $b^1 = b^2 = 1$ so that he acts as a buyer on the most expensive post. Intuitively, he could switch his unit of account from post 1 to post 2, in order to improve his final allocation. In this example it is not an optimal strategy. If agent h decides to switch a unit of account from post 1 to post 2, the prices become $(p^1)' = (p^2)' = 5$, and the final quantity of good he gets from the market is smaller, $2/5 < 5/12$.

This impossibility to take advantage of the price difference is based on two contradicting influences. The first lies in the price difference between both posts, which is the usual argument explaining why arbitrage behaviors lead to the uniformity of prices. With the extra unit, the agent gets $1/5$ unit of good on post 2 instead of the $1/6$ unit post 1, so that the net gain of the trade off is $1/30 > 0$. The second effect, the one we refer to as the *marginal unit* effect, runs as follows. Any agent changing his strategy on a post modifies the price, such that this behavior affects the entire allocation gotten from this post and not only the one gotten with the marginal bid. The first unit of account deposited on post 2 buys now $1/5$ unit of commodity instead of the $1/4$ unit before the transfer, so that the value of the negative *marginal unit* effect is $-1/20 < 0$. Finally, summing up both effects—the net value is $-1/60$—the gain associated to the initial price difference is not enough to compensate the negative *marginal unit* effect on the second post: the arbitrage opportunity turns out to be non-profitable.

Let us mention here this striking situation to occur needs a specific condition to be satisfied: agents have a relatively higher level of activity on less expensive posts—$b^1/B^1 < b^2/B^2$. It is worth noticing this is a necessary condition, and that the size of the difference between relative weights that ensures a strategy to be optimal depends on the importance of the price difference. Consider a variant of the previous example where the *relative weight* condition fails, such as $b^1 = 2$ and $B^1 = 6$. This situation exhibits a profitable arbitrage opportunity.

A first intuition on the influence of wash sales. The impact of wash-sales is of different natures, depending on whether it applies to the agent trading them or to others. We focus here on this second case. As wash-sales do not change the payoff of the agent trading them, the link between wash-sales and price dispersion has to do with the impact of wash-sales on replies of others. To be short, they lower the negative influence other agents have on prices. The intuition is the following. As we highlighted through the exposition of the marginal unit effect, any agent, by his own action, modifies the market price in the negative—He buys, the price increases, He sells, the price decreases. Now, wash-sales limit this negative influence, due to the fact that they increase the thickness of the post.

Let us picture that idea within an example. Assume an agent $k \neq h \in H$ trades wash-sales on post 1 when agent h is bidding 3 units of account, given that initially $B^1 - b^1 = 5$ and $B^2 - b^2 = 3$. We know that $(b^1, b^2) = (1, 2)$ is a better strategy than $(2, 1)$. The situation is different if agent $k \neq h$ plays wash-sales on post 1. Assume $(b^1, b^2) = (2, 1)$, implying $p^1 = 7$ and the allocation is $15/28$. Consider wash-sales of agent k which value corresponds to 2 units of good, implying $B^1 = 19$ and $Q^1 = 3$. Naturally, the price and allocations are unchanged. Now, if agent h transfers one unit from post 2 to post 1, he gets $1/2 < 15/28$, such that wash-sales on post 1 give an incentive to agent h to play the two units on the most expensive post.

3.2 A Two-posts market

Consider now the general case and distinguish two cases, $q^1q^2 = 0$ and $q^1q^2 > 0$.

First case: $q^1q^2 = 0$**. Post 1.** When an agent deposits $(b^1 - a) > 0$ units with $a > 0$ instead of b^1 units of account on post 1, the quantity of good bought on this post is diminished from v_c^1 units:

$$\begin{aligned} v_c^1(a) &= \left[\left(b^1 - a\right) Q^1\right] / \left(B^1 - a\right) - b^1 Q^1 / B^1 \\ &= -a \cdot \left[Q^1 / B^1\right] \left[\left(B^1 - b^1\right) / \left(B^1 - a\right)\right] < 0 \end{aligned} \tag{6}$$

If the agent is not selling on this post, he finally gets $v_m^1 = a$ more units of account.

Post 2. If the agent increases his bid b^2 of $c > 0$ units on post 2, this modification implies a variation in the quantity of good equal to v_c^2:

$$v_c^2(c) = \left(b^2 + c\right) Q^2 / \left(B^2 + c\right) - b^2 Q^2 / B^2$$
$$= c \cdot \left[Q^2 \left(B^2 - b^2\right)\right] / \left[B^2 \left(B^2 + c\right)\right] > 0 \quad (7)$$

Transfer of "a" units of account. Using equations (6) and (7), switching a units of money from post 1 to post 2 is not interesting if $v_c^1(a) + v_c^2(a) \leq 0$. When a goes to zero, this condition can be rewritten as:

$$p^1/p^2 \leq \left[\left(B^2\right)\left(B^1 - b^1\right)\right] / \left(B^1\right)\left(B^2 - b^2\right) \quad (8)$$

Condition (8) defines the set of situations with price dispersion that preclude arbitrage opportunities to be profitable.

Entering interesting posts first. Notice that when $p^1 > p^2$, specification (8) can equivalently rewritten as the *relative weight* condition

$$b^2/B^2 > b^1/B^1. \quad (9)$$

Therefore, an agent enters first the cheapest post, and his activity on this post is more important than the one on the other one.

Not to cannibalize a trading-post. The arbitrage condition is Eq. (8) when equality holds. Given the definition of p^1 and p^2, it is easy to rewrite this condition as

$$\frac{B^1}{B^1 - b^1}\frac{B^1}{Q^1} = \frac{B^2}{B^2 - b^2}\frac{B^2}{Q^2} \quad (10)$$

Consider Eq. (10). For equality to hold when $p^1 > p^2$, as b^1 in included in B^1 and b^2 is included in B^2, we first need b^2 to increase. However, once b^2 exceeds a given value, b^1 needs to increase too—even if the gradient is less important. Hence, agent h enters a post with a less interesting price once his weight on the market becomes too large, that is once his relative weight on an interesting post exceeds a critical value. As a result, agents have interest not to be too large on a given post. This suggests the following interpretation about the role of liquidity constraints.

Liquidity constraints. [8] An agent can promote the dispersion of prices only if he trade on the post with the less interesting price. But, any agent enters first the post with the most interesting price and starts entering the less interesting

[8] We consider *money* constraints on purpose. In this way, we give an intuition of why the results formally stated in the paper for *commodity* constraint extend to money liquidity constraints.

ones only once his relative weight exceeds a critical value. Nonetheless, even in that case he keeps intervening much more on the price interesting post—so as to satisfy the relative weight condition—so that the price dispersion is continually reduced. As a result, liquidity constraints may prevent agents from trading off a price dispersion only. They cannot entirely explain a dispersion of prices.

Second case: $q^1 q^2 > 0$**.** The impact of $q^1 > 0$ is to introduce a new marginal unit effect on post 1. The transfer of $a > 0$ units alters the value of the sales of ${}_{ws}v_m^1$ unit:

$$ {}_{ws}v_m^1 (a) = q^1 \left(B^1 - a\right) / Q^1 + a - q^1 B^1 / Q^1 = a \left(1 - q^1 / Q^1\right) \tag{11} $$

Hence, as ${}_{ws}v_m^1 (a) < a$ if $q^1 > 0$, when he takes away a units of account from post 1 the agent can only add $c = a \left(1 - q^1/Q^1\right) < a$ unit on post 2, that is the amount a which takes into account the *marginal unit* effect associated to $q^1 > 0$.

A similar effect occurs when $q^2 > 0$. Because of $q^2 > 0$, the transfer alters the value of the sales on post 2 of $a \left(1 - q^1/Q^1\right) q^2/Q^2$. The budget constraint keeps binding if, when he takes off a units of account from post 1, agent h adds $a \left(1 - q^1/Q^1\right) q^2 / \left(Q^2 - q^2\right)$ units on post 2.[9] In that case, the final amount of commodity ${}_{wss}v_c (a)$ resulting from the money transfer is:

$$ {}_{wss}v_c (a) = v_c^1 (a) + v_c^2 \left(a \left[\left(Q^1 - q^1\right) / Q^1\right] \left[q^2 / \left(Q^2 - q^2\right)\right]\right) $$

and agent h cannot take advantage of the price difference when this quantity is negative. That is, when transfers are marginal:

$$ p^1/p^2 \leq \left[Q^1 \left(Q^2 - q^2\right)\right] / \left[Q^2 \left(Q^1 - q^1\right)\right] \left[B^2 \left(B^1 - b^1\right)\right] / \left[B^1 \left(B^2 - b^2\right)\right] \tag{12} $$

Equation (12) defines the new set of strategies and prices excluding the possibility to take advantage of the price difference. It is worth noticing this is the condition (8) with the multiplying factor $\left[\left(Q^2 - q^2\right) Q^1\right] / \left[Q^2 \left(Q^1 - q^1\right)\right]$, which can be rewritten as a relative weight condition expression. Finally, the no-arbitrage condition is Eq. (12) when equality holds.

Influence of wash-sales of others. Consider the general case, where condition (12) applies, and assume agent $k \neq h$ trades wash-sales for an amount

[9] In that case we have that the net monetary value following the transfer and including bids and offers on all of the sides is $0 = +a$ (purchases on post 1) $-aq^1/Q^1$ (sales on post 1) $-a \left(1 - q^1/Q^1\right) \left(q^2 / \left(Q^2 - q^2\right)\right)$ (bid on post 2) $+a \left(1 - q^1/Q^1\right) \left(q^2 \left(Q^2 - q^2\right)\right) q^2/Q^2$ (offer on post 2).

of ws_k^s units on post s. Focusing on situations when agent h is only bidding for commodities, $b_h^1 > 0$, $q_h^1 = 0$, $b_h^2 > 0$ and $q_h^2 = 0$, after straightforward manipulations—given that prices do not change—condition (12) at equality can be rewritten as:

$$\left[1 - \frac{b^1}{B^1 + ws_k^1}\right] \frac{Q^1}{B^1} = \left[1 - \frac{b^2}{B^2 + ws_k^2}\right] \frac{Q^2}{B^2} \tag{13}$$

Therefore, increasing wash-sales on a post implies the agent increases his bid on this post, or decreases his bid on the other one. When $p^1 > p^2$, we observe that, increasing wash-sales on post 1 and/or reducing wash-sales on post 2 enlarges the difference between b^1 and b^2 which satisfy condition (13). Logically, when wash-sales are added on the most expensive post, the negative effect on price p^1 from bidding b^1 is less important, and agent h has incentives to offer a greater amount of money on the most expensive post. A similar effect appears if agent k diminishes his amount of wash-sales on post 2.

4 Equilibrium characteristics and arbitrage

In the way of analyzing individual behaviors, if we want to include situations with endowment constraint binding we cannot follow Koutsougeras (2003*b*) whose analysis hinges on differentiable functions of outcomes. So, we explore and solve individual maximizing programs.

4.1 Equilibrium characteristics

It is well known that utility functions are not necessary concave in bids and offers. To circumvent this difficulty maximizing problems are defined over the consumption set (Peck, Shell and Spear (1992)). But, situations with endowment constraints binding adds another problem. Non-negativity bidding constraints do not necessary define a convex subset in the allocation space, and non-negativity constraints are binding too if an endowment constraint is binding (see below). The reason is that, given the price formation rule, a marginal decrease of an offer is equivalent to a marginal increase of the bid on the same post (apart from a (price) multiplying factor). Therefore, in a first step we rewrite the program as a convex program, using the symmetry property between bids and offers.

First, given the multiplicity of trading-posts it is easier to expose the results using the following notation and concept. Assume i's endowment constraint is binding—the Lagrange multiplier λ_h^i is negative. If prices of good i are not uniform, the agent can be "*effectively constrained*" by his i's endowment on *several* trading-posts *only*. What is meant here is that the cost λ_h^i is not necessary sustained by the agent h on posts (i,r) where the price is not high enough. There may be (active) posts (i,r) where the endowment constraint is not *effectively* precluding the agent to offer more commodity—in which case he does not offer at all on this post.

Definition 5 *If λ_h^i is the multiplier associated to the commodity i endowment constraint, we denote $\lambda_h^{i,s}$ such that: $\lambda_h^{i,s} = \lambda_h^i$ if the constraint $\sum_{s=1}^{K_i} q_h^{i,s} \leq e_h^i$ binding* "effectively constraints" *the agent h to increase his i's offer on post (i,s), and $\lambda_h^{i,s} = 0$ otherwise.*

The interest in this definition is to define the necessary and sufficient conditions under the form they are expressed when there is only one post for each commodity type.

Proposition 6 *Consider the utility maximization problem of agent $h \in H$. Given strictly positive bids $B_h^{i,s} > 0$ and offers $Q_h^{i,s} > 0$, a strategy $\sigma_h \in S_h$ is optimal if and only if:*

$$\partial u_h / \partial x_h^i = -\gamma_h \left(p^{i,s}\right)^2 Q_h^{i,s} / B_h^{i,s} + \lambda_h^{i,s} \tag{14}$$

for each (i,s), with γ_h the multipliers associated to the budget constraints.

We proceed by proving intermediate results. Among them, some are presented in the body text and the others, as the demonstrations, are relegated to the Appendix.

Lemma 7 *For each $i = 1, ..., L$, if the endowment constraint $\sum_{s=1}^{K_i} q_h^{i,s} \leq e_h^i$ is binding, either 1. $\lambda_h^{i,s} = \lambda_h^i < 0$, and $\lambda_h^i = p^{i,s}\beta_h^{i,s}$ with $\alpha_h^{i,s} = 0$—non negativity constraint $b_h^{i,s} \geq 0$ is binding, or 2. $\lambda_h^{i,s} = 0$, with $\alpha_h^{i,s} = \lambda_h^i$ and $\beta_h^{i,s} = 0$—non negativity constraint $q_h^{i,s} \geq 0$ is binding.*

The idea underlying this result is the following. Consider an agent h whose i's endowment constraint is binding, i.e. $\lambda_h^i \neq 0$. 1. Agent h has two possibilities to get more money on a post (i,s): (i) offering more commodity i (ii) bidding a negative amount of money on (i,s). In accordance with the definition of the allocation rule, an additive marginal offer of good is similar to a marginal subtraction of the bid for this good. As a consequence, the costs of both constraints are the same. 2. A similar reasoning can be made for constraints $q_h^{i,r} \geq 0$. We know agent h wants to sell more of commodity i. But, it may be that he does not want to sell more of i on posts (i,r) where the price is not high enough. Logically, he plays $q_h^{i,r}=0$ on those trading-posts. Nevertheless, agent h could relax his endowment constraint $\sum_{s=1}^{K_i} q_h^{i,s} \leq e_h^i$ binding by adding a negative term to the sum, if he plays $q_h^{i,r}<0$. Therefore, on those posts (i,r) the constraint $q_h^{i,r} \geq 0$ is binding and $\alpha_h^{i,r}=\lambda_h^i$, because the agent has an incentive to play $q_h^{i,s}<0$.

Now, this formal result implies that there is an equivalence between first order conditions in respect to $b_h^{i,s}$ and $q_h^{i,s}$.

Proposition 8 *$\forall\ i=1,...,L$, for any active post (i,s), the first order conditions in respect to the bid strategies are redundant to the first order conditions in respect to the offer ones.*

This symmetry property, about bid and offer strategies, implies that the maximizing program can be defined as a convex program on the allocation space X. It is indeed sufficient to consider constraints $\sum_{s=1}^{K_i} q_h^{i,s} \leq e_h^i$ and $q_h^{i,s} \geq 0$ (defined on X), so that the program is convex. Therefore, there is a unique optimum which necessary and sufficient conditions are characterized by solving this program (see appendix).

Further, by way of simplicity we are focusing on the optimal behavior of an isolated agent when he does not trade wash-sales. In this respect, by demonstrating that wash-sales do not impact on the best reply allocation of the agent we want to legitimate the generality of these results.

Proposition 9 *In the Bid or Sell market game, given moves of others as given and strictly positive, a strategy $\sigma_h \in H$ is optimal if and only if:*

$$\begin{aligned} \partial u_h / \partial x_h^i &= -\gamma_h \left(p^{i,s}\right)^2 Q_h^{i,s} / B_h^{i,s} + \lambda_h^{i,s} \\ 0 &= b_h^{i,s} \cdot q_h^{i,s} \end{aligned}$$

for each post (i,s), with γ_h the multiplier associated to the budget constraint.

The proof hinges on the fact that wash-sales do neither alter allocations nor prices. This suggests the following observations. First, an equilibrium without wash-sales is an equilibrium in both the Bid or Sell and the Bid - Offer models. Then, first order conditions are similar that some wash-sales are played or not. In consequence, looking at the strategy of an agent with or without wash-sales does not matter and the following general no-arbitrage conditions, which will be used below, can be stated that agent h places wash-sales or not.

Proposition 10 *Given a strictly positive set of moves from others $s_{-h} \in S_{-h}$,* *the no-arbitrage requirement of an optimizing agent h satisfies,* $\forall$ *i, s, r:*

$$\left(p^{i,s}\right)^2 Q_h^{i,s}/B_h^{i,s} - \lambda_h^{i,s} = \left(p^{i,r}\right)^2 Q_h^{i,r}/B_h^{i,r} - \lambda_h^{i,r} \tag{15}$$

It is notworthy that specification (15) is the no-arbitrage rule that can be found in Koutsougeras (2003*b*) when the set of agents facing liquidity constraint is empty (in which case $\lambda_h^{i,s} = \lambda_h^{i,r} = 0$).

4.2 Non-profitable arbitrage opportunities

This section generalizes results stated in the two-posts example. As the different principles have already been fully exposed, the results are simply presented in their general form. In what follows, we consider the strategy without wash-sales, but results can easily be rewritten in terms of net trade when wash-sales. [10]

Proposition 11 *Let two posts where a commodity type* $i = 1, ..., L$ *is exchanged at strictly positive prices be* (i,s) *and* (i,r)*. When* $p^{i,s} > p^{i,r}$*, if the strategy* $\sigma_h \in S_h$ *without wash-sales is optimal we have*

(i) An agent entering the buying (selling) side of a post never offers (bids) on other posts being less price interesting, necessary bids (offers) on all other more price interesting posts and satisfies the relative weight condition $b^{i,s}/B^{i,s} < b_h^{i,r}/B^{i,r}$ $(q_h^{i,s}/Q^{i,s} > q_h^{i,r}/Q^{i,r})$*,*

(ii) if the agent is a (net) buyer on post i, r *and a (net) seller on post* i, s*:*

$$\left(p^{i,s}\right) Q_h^{i,s}/Q^{i,s} - \lambda_h^{i,s} = \left(p^{i,r}\right) B^{i,r}/B_h^{i,r} \tag{16}$$

(iii) if the agent is a (net) buyer on both posts:

$$p^{i,s} B^{i,s}/B_h^{i,s} = p^{i,r} B^{i,r}/B_h^{i,r} \tag{17}$$

(iv) if the agent is a (net) seller on both posts

$$p^{i,s} Q_h^{i,s}/Q^{i,s} = p^{i,r} Q_h^{i,r}/Q^{i,r} \tag{18}$$

[10] Koutsougeras (2003*b*) deduces price dispersion from specific conditions on net trades of agents. Here, we deduce some characteristics of agents' net trade from conditions on the dispersion of prices. It is noteworthy all results of Koutsougeras apply when $\exists\ i$ s.t. $C_i(b,q) \neq \varnothing$.

(v) *σ_h satisfies condition [20] when he his a buyer, so that he enters the cheapest post first, and enters the post with a higher price once his weight on the other exceeds a critical value.*

Proof. Let two posts where a commodity $i = 1, ..., L$ is exchanged at a strictly positive price be (i, s) and (i, r) $(r,\ s = 1, ..., K^i)$. We know if the strategy $\sigma_h \in S_h$ is optimal Eq. (15) is satisfied.

(i) Suppose $b_h^{i,s} > 0$. Assume $b_h^{i,r} = 0$ and $q_h^{i,s} = 0$, and $q_h^{i,r} b_h^{i,s} > 0$. We have $B_h^{i,r} = B^{i,r}$ and $Q_h^{i,s} = Q^{i,s}$, so that if $p^{i,s} > p^{i,r}$, $B^{i,s}/B_h^{i,s} - \lambda_h^{i,s} > 1 \geq Q_h^{i,r}/Q^{i,r}$ implies the no-arbitrage condition cannot be satisfied. Suppose now $q^{i,r} > 0$ and $q_h^{i,s} = 0$. We have $b_h^{i,r} = 0$, $B_h^{i,r} = B^{i,r}$ and $Q_h^{i,s} = Q^{i,s}$. By the no-arbitrage requirement we can assert that:

$$\left(p^{i,r}\right) B^{i,r}/B_h^{i,r} - \lambda_h^{i,r} = \left(p^{i,s}\right) Q_h^{i,s}/Q^{i,s}$$

which can be checked when $p^{i,s} > p^{i,r}$. Now, suppose agent h is bidding on post (i, s). We know $q^{i,r} = 0$. Hence, we must have $\lambda_h^{i,s} = \lambda_h^{i,r} = 0$, $q_h^{i,s} = q_h^{i,r} = 0$ and $Q_h^{i,r} = Q^{i,r}$ and $Q_h^{i,s} = Q^{i,s}$. This implies that when $p^{i,s} > p^{i,r}$, from the no-arbitrage condition we have $B^{i,s}/B_h^{i,s} < B^{i,r}/B_h^{i,r}$, or equivalently $b_h^{i,s}/B^{i,s} < b_h^{i,r}/B^{i,r}$. Moreover, if $b_h^{i,s} > 0$ we have $b_h^{i,r} > 0$. A symmetric analysis proves the condition $q_h^{i,s}/Q^{i,s} > q_h^{i,r}/Q^{i,r}$, obtained when $b_h^{i,s} = b_h^{i,r} = 0$.

Hence, $p^{i,s} > p^{i,r}$ implies that the situation when an agent buys on the most expensive post and sell on the less expensive one cannot be a best reply

Therefore, proof of (ii), (iii) and (iv) follows immediately from the no-arbitrage conditions.

(v) Eq. (20) follows from the no-arbitrage conditions. Assume $b_h^{i,s} = 0$ and $b_h^{i,r} > 0$. If $p^{i,r} < p^{i,s}$, agent h plays $b_h^{i,r} > 0$ in order to satisfy

$$Q_h^{i,r} B_h^{i,r} B_h^{i,s}/Q_h^{i,s} = \left[B_h^{i,r} + b_h^{i,r}\right]^2 \tag{19}$$

Once $b_h^{i,r}$ is great enough to satisfy (19), if agent h wants to buy more of the commodity i, he needs to enter the post (i, s) too. Otherwise, condition (20) cannot be satisfied. Then, agent h increases $b_h^{i,r}$ and $b_h^{i,s}$ so as to satisfy this last equation. **End proof**

This proposition provides a complete exposition of the no-arbitrage conditions. Then, the reason why an agent intervenes on a post with a less interesting price can be identified from the no arbitrage conditions, as far as these conditions are rewritten correctly as in the example of the previous section

$$\frac{B_h^{i,s} + b_h^{i,s}}{Q_h^{i,s}} \frac{B_h^{i,s} + b_h^{i,s}}{B_h^{i,s}} = \frac{B_h^{i,r} + b_h^{i,r}}{Q_h^{i,r}} \frac{B_h^{i,r} + b_h^{i,r}}{B_h^{i,r}}. \tag{20}$$

Eq. (20) shows that a bidder for a commodity i intervenes on the cheaper post first, and intervene on the other one once his volume of trade is large enough. We explore this idea more precisely in the next section in case agent h is a net seller, in order to connect the results to the effects of liquidity constraints.

5 Price dispersion and liquidity effects

This sections explains why liquidity constraints are not critical for the dispersion of prices. Let us first establish the following intuitive result. Denote $p_h^{i,s} = B_h^{i,s}/Q_h^{i,s}$ the price formated on post (i, s) if agent h does not play on this post, $\forall\, i, s$ and h, and let us show an agent may diminish price dispersion only.

Proposition 12 *Considering moves of others as given, the optimal behavior of an agent may only reduce the dispersion of prices, in the sense that if* $p_h^{i,s} p_h^{i,r} > 0$ *and* $p_h^{i,s}/p_h^{i,r} \geq 1$ *we have* $p_h^{i,s}/p_h^{i,r} \geq p^{i,s}/p^{i,r}$.

Proof. Consider two active posts (i, s) and (i, r) satisfying $p_h^{i,s} > p_h^{i,r}$. Without lost of generality, let us consider h's best-replies without wash-sales. First, we know that if agent h is a buyer on a post and a seller on another one he will necessary restrict the price difference (proposition (12)). Now, assume agent h is a seller on both posts (the case he is a buyer is symmetric). From condition (15), his best reply satisfies:

$$p^{i,s}/p^{i,r} = \sqrt{Q_h^{i,r} B_h^{i,s} / B_h^{i,r} Q_h^{i,s}} = \sqrt{p_h^{i,s}/p_h^{i,r}} \tag{21}$$

implying that the price dispersion between both posts is reduced, as $p_h^{i,s}/p_h^{i,r} \geq 1$. **End proof**

This result is intuitive in the sense one expects an agent does his best to take advantage of a dispersion of prices, and by the way reduces it. It also underlines the fact that price dispersion is the result of an equilibrium condition, as the price uniformity in the Bid or Sell market game (Gobillard, 2006). This result does however not take into account the impact of any behavior on replies of others, so that one must be cautious in interpreting it.

Now, let us turn our attention to the role of liquidity constraints. It can first be deduced from proposition (12). If an agent never increases the price dispersion, he wont generate price dispersion even if his endowment constraint is binding. Eventually, this constraint may limit his ability to trade off prices as much as he would like to. A more detailed explanation runs as follows. Consider the situation without wash-sales and assume agent h is a seller of commodity i and (i, s) is the most expensive post. So that the agent generates/increases price dispersion he must enter less expensive posts $(i, r) \neq (i, s)$ as a seller. Now, the relative weight condition $q_h^{i,s}/Q^{i,s} > q_h^{i,r}/Q^{i,r}$ implies agent h will always enter the most expensive post first. He will enter a less expensive post (i, r) only when his relative weight on post (i, s) becomes too large. The proposal is therefore obvious if the agent faces a commodity constraint until then. But, even if he faces a liquidity constraint once he has entered both posts, the relative weight condition ensures that for each unit deposited on post (i, r) a large enough amount of commodity is deposited on post (i, s) such that the price difference keeps reducing.

This idea is written down in the following proposition. By definition, when $p_h^{i,s} > p_h^{i,r}$, if agent h does not enter posts (i, r) and (i, s) we have $p^{i,s} > p^{i,r}$. Now, consider $b_h^{i,s} = b_h^{i,r} = 0$. When $p_h^{i,s}/p_h^{i,r} > 1$, let $\hat{q}_h^{i,s}$ be the critical value defined by

$$\hat{q}_h^{i,s} = Q_h^{i,r} \left(\sqrt{p_h^{i,s}/p_h^{i,r}} - 1 \right) > 0 \tag{22}$$

Proposition 13 *Suppose $p_h^{i,s}/p_h^{i,r} > 1$ and $b_h^{i,r} = b_h^{i,s} = 0$. If agent h enters post (i, s) as a seller, his strategy profile satisfy*

$$q_h^{i,r} \left(q_h^{i,s} \right) = \left[1/\sqrt{p_h^{i,s}/p_h^{i,r}} \right] \; q_h^{i,s} - Q_h^{i,r} \left[1 - 1/\sqrt{p_h^{i,s}/p_h^{i,r}} \right] < q_h^{i,s} \tag{23}$$

once $q_h^{i,s} > \hat{q}_h^{i,s}$ and 0 otherwise, and the price ratio function in respect to $q_h^{i,s}$, $f\left(q^{i,s}\right) = \left[p^{i,s}/p^{i,r}\right]$, is a decreasing function of $q^{i,s}$.

Proof. By proposition (11), when $p^{i,s} > p^{i,r}$, if $q_h^{i,r} > 0$ we have $q_h^{i,s} > 0$ and $\lambda_h^{i,s} = \lambda_h^{i,r}$. Therefore, in these cases, Eq. (15) can be rewritten as

$$\left[B_h^{i,s} Q_h^{i,r} / B_h^{i,r} Q_h^{i,s} \right] \left(Q_h^{i,r} + q_h^{i,r} \right)^2 = \left(Q_h^{i,s} + q_h^{i,s} \right)^2 \tag{24}$$

and with simple calculations Eq. (23) is obtained.

Now, it is noteworthy that, as $p_h^{i,s}/p_h^{i,r} > 1$, we have $\hat{q}_h^{i,s} > 0$ and so $q_h^{i,s} > q_h^{i,r}$. Then, it becomes obvious that $\hat{q}_h^{i,s}$ is the limit value of $q_h^{i,s}$ until $q^{i,r}$ becomes strictly positive. When the strategy $q_h^{i,s}$ is inferior to $\hat{q}_h^{i,s}$ because of liquidity constraints binding or because it is the best reply regardless to the amount of the initial endowment, we must have $q_h^{i,r} = 0$.

Therefore, it is easy to show that:

$$\begin{aligned}\left[B_h^{i,s}/B_h^{i,r}\right] f\left(q^{i,s}\right) &= \left[p^{i,s}/p^{i,r}\right]\left[B_h^{i,s}/B_h^{i,r}\right] = \left[Q_h^{i,r} + q_h^{i,r}\right] / \left[Q_h^{i,s} + q_h^{i,s}\right] \\ &= \left[1/\sqrt{p_h^{i,s}/p_h^{i,r}}\right]\left[q_h^{i,s} + Q_h^{i,r}\right] / \left[q_h^{i,s} + Q_h^{i,s}\right]\end{aligned}$$

is a decreasing function of $q_h^{i,s}$, as $\sqrt{p_h^{i,s}/p_h^{i,r}} > 1$. **End Proof**

These analyses result in the following general proposition.

Proposition 14 *Liquidity constraints do not generate price dispersion per se, and may only limit the ability of agents to trade off prices.*

The proof follows from either proposition (14) or proposition (13).

The above analysis sheds some light on the fact that the role of liquidity constraints may be of different nature in the commodity pairwise market game. The distinction lies in the existence of a unique medium of exchanges. So, an agent trading a good for another one is always able to use the optimal vector of transactions—to achieve his plan—within the money market game, in the sense that he is able to enter all the trading-posts where these commodities are traded. To exchange x for y, intermediate trades $x - money$ and $money - y$ are necessary. Therefore, either the agent can sell x (buy y) on all posts where x is sold (y is bought) or he cannot sell x (buy y) on any trading-post. This property is no more true if commodities exchange for commodities, in which case transactions may be under the form $x - z$ and $z - y$ (for any z) and the agent needs to be furnished in his 'intermediate' good z. This has the following consequences. When prices are not consistent, the optimal vector of transactions imposes the agent to trade on given trading-posts so that intermediate transactions can be required. In this case, if the agent is not correctly furnished in these (intermediate) commodities he cannot make those trades and ends up intervening on other trading-posts which increases the price inconsistency. In other words, an agent not furnished enough in some goods may be unable to intervene on posts with most interesting prices, so that he plays strategies inducing dispersion of prices. This phenomenon disappear if money is used in trade.

6 Convergence to price uniformity

We now turn to the issue of the convergence to price uniformity in the limit. The question is the following: What does become price dispersion as the number of agents increases? Our purpose is to show it goes to zero even when there are liquidity considerations. More precisely, the law of one price for i holds when the number of agents trading i increases and a sufficient condition is that the number of agents endowed with i grows. Therefore, this result follows the line of the previous sections.

Consider an equilibrium profile $\sigma \in S$ that achieves a non-uniform set of prices and assume post 1 is the most expensive post. For commodity i, let T^i define the number of active posts and $g^i(\sigma)$ be a measure of the larger price differential:

$$g^i(\sigma) = \underset{p^{i,s} \in p^i}{Max} \left(\left[p^{i,s}/p^1 \right] - 1 \right)$$

Let $\mathcal{E}_n$ be a sequence of economies with $\mathbf{k}_n$ representing their market structures and where $\# H_n \to \infty$. Let $(\sigma_n) \in S^{\mathbf{k}_n}$ be a sequence of equilibria and define $z_n^i = \# H_n / T_n^i + 1$. Finally, denote

$$C_n^i(b,q) = \left\{ h \in H_n : \sum_{i=1}^{K^i} q_h^{i,s} = e^i, b^{i,s} q^{i,s} = 0 \right\}$$

and

$$U_n^i(b,q) = H_n \backslash C_i^n(b,q)$$

the sets of unconstrained/constrained agents.

Then, it can be found that as far as $C_i^n(b,q) = \emptyset$ the price dispersion vanishes in the limit (Koutsougeras, 2003*b*). This result is written down in the following theorem.

Theorem 15 (Koutsougeras, *2003b*) *If $C_i^n(b,q) = \varnothing$, $z_n^i \to \infty \Rightarrow g^i(\sigma_n) \to 0$.*

Now, taking into account liquidity constraints—that is $C_i^n(b,q) \neq \varnothing$, as the number of agents whose i's endowment constraint is not binding increases, the price dispersion for commodity i vanishes. This result is stated in the following theorem. [11]

[11] While surprisingly it is not explicitely formulated in Koutsougeras' paper, it can be deduced from his analysis and actually appears as to be clearly suggested. However, a complete proof follows from our results stated hereafter.

Theorem 16 $\#U_n^i(b,q)/T_n^i \to \infty \Rightarrow g^i(\sigma_n) \to 0$.

Our purpose here is to show that what is crucial for this result is the number of agents trading a commodity, in the following sense. Increasing the number of agents trading commodity i is sufficient for the convergence to price uniformity in p^i, and this number of agents increases quicker than $\#U_n^i(b,q)$. Indeed, by definition of $C_i^n(b,q)$, if $C_i^n(b,q) = \emptyset$ any agent is endowed with each commodity, $e_h^i > 0$, $\forall\, i$, h. Now, it can be shown that when the law of one price is violated for a commodity i, all agents endowed with this commodity exchange it (theorem 17). Therefore, if increasing the number of unconstrained agents makes price dispersion vanish, it is rather because in this case the number of agents trading (endowed with) this commodity increases when prices are different.

Let us denote

$$E_n^i = \left\{h \in H : e_h^i > 0\right\} \subset H_n$$

the number of agents endowed with the commodity i, and

$$A_n^i = \left\{h \in H : \exists\ s = 1, ..., K^i,\ b_h^{i,s} \text{ or } q_h^{i,s} > 0\right\} \subset H_n$$

the number of agents trading the commodity i. We first state the following theorem.

Theorem 17 $\forall\ i = 1, ..., L$ *such that* p^i *violates the law of one price, we have* $E_n^i \subseteq A_n^i$.

Proof. Assume commodity i is exchanged at different prices. The proof is to show any agent $h \in E_n^i$ will trade commodity i. Consider an agent h such that $e_h^i > 0$. First, we know that if the endowment constraint of agent h is binding he is trading commodity i. Then, assume the endowment constraint of agent h is not binding ($\lambda_h^i = 0$). The no-arbitrage condition implies, $\forall\ r$ and s (which both post strictly positive prices),

$$\left(p^{i,s}\right)^2 Q_h^{i,s}/B_h^{i,s} = \left(p^{i,r}\right)^2 Q_h^{i,r}/B_h^{i,r}$$

Then, when $p^{i,s} < p^{i,r}$, if agent h does not trade commodity i, we must have $1 = p^{i,s}Q_h^{i,s}/B_h^{i,s} < p^{i,r}Q_h^{i,r}/B_h^{i,r} = 1$, a contradiction. **End Proof.**

It is quite intuitive that any agent endowed with a commodity i will trade this commodity type as far as positive prices for this commodity are not uniform. This means agents have interest in arbitraging (even partially) any price difference as far as they are able to, and a sufficient conditions is to be endowed with this commodity. Therefore, when $C_i^n(b,q) = \emptyset$ all agents trade commodity i.

Then, it is possible to show that the convergence result applies whatever is the number of agents whose liquidity constraint is binding, as far as $\#A_n^i/T^i$—and therefore $\#E_n^i/T^i$—goes to zero.

Theorem 18 $\#E_n^i/(T^i+1) \to \infty \Rightarrow g^i(\sigma_n) \to 0.$

The proof is a direct application of the theorems (17) and (19).

Theorem 19 $\#A_n^i/(T^i+1) \to \infty \Rightarrow g^i(\sigma_n) \to 0.$

The proof is relegated to the appendix.

So, generically price dispersion vanishes as the number of agents grows. The main conclusion that can be deduced from these results is that convergence to price uniformity does not have so much to do with agents not to be constrained by their commodity endowment. The convergence to the law of one price is explicitly linked to the number of agents trading any given commodity. In that sense, this property can be understood in the light of the level of activity agents have on the market, more than through the lenses of the number of agents—because unconstrained—able to arbitrage prices as much as they would like to.

7 On the market structure

To conclude this contribution, we analyze the influence of the structure of markets. Is is known that the law of one price holds if wash-sales are precluded (Gobillard, 2006). Therefore, it is natural to assess the value of adding trading-posts to the canonical market-game. So, we compare allocation sets when the number of trading-posts evolves, which means we examine when an allocation achieved in the multiple trading-posts market game can/cannot be achieved by an equilibrium in the canonical market game. In some sense we generalize results stated by Koutsougeras (2003*b*) [12] . Actually, our analysis of the market game allows to examine situations with liquidity considerations in which case

[12] Similar results are obtained by Bloch and Ferrer (2001) in the context of a bilateral oligopoly.

$\exists\, i \in 1, ..., L$ s.t. $C_i\,(b, q) \neq 0.$[13] Mainly, we believe the relevance of using the market game with only one trading-post is not really weakened by the analyses of the market game with multiple trading-posts. In the following, as results apply in case wash-sales are allowed or precluded, we wont specify this point.

Let $\mathbf{E}^{\kappa}$ be a market game where κ is the L-dimensional vector defining the strictly positive number of posts:

$$\kappa = \left\{K^i : K^i > 0,\, i = 1, ..., L\right\}$$

Define S_h^{κ} as the feasible set of strategy profiles relative to the market game $\mathbf{E}^{\kappa}$. Let τ be a L-dimensional vector taking strictly positive integer values:

$$\tau = \left\{T^i : T^i > 0,\, i = 1, ..., L\right\} \tag{25}$$

We say that $\tau \geq \kappa$ if, for each commodity i, we have $T^i \geq K^i$.

Finally, let (p, x) be a set of allocations and prices, and τ a $L-$dimensional vector of strictly positive numbers. We define $\mathbf{A}^{\tau}$ as the set of equilibrium allocations (p, x) of the market game $\mathbf{E}^{\tau}$.

The study of this section leads to the conclusion that increasing the number of trading posts may enlarge the set of equilibrium allocations with allocations that contravene the law of one price only, and that the sets of allocations which satisfy the law of one price are not affected by the addition/subtraction of trading-posts.

Proposition 20 *We have (i) $\mathbf{A}^{\tau} \subseteq \mathbf{A}^{\kappa}$ whenever $\tau < \kappa$; (ii) Allocations and prices $(x, p) \in \mathbf{A}^{\kappa} - \mathbf{A}^{\mathbf{1}}$ violate the law of one price whatever is $\mathbf{0} < \kappa$; (iii) Whatever are the numbers of posts $\kappa > \mathbf{0}$ we have $\mathbf{A}^{\mathbf{1}} = \mathbf{A}^{\kappa}$ in in the Bid or Sell market game and for equilibrium satisfying the law of one price.*

The proof is relegated to the appendix. It hinges on two intermediate results. First, an equilibrium allocation of a given market game can still be achieved by an equilibrium if the number of trading-posts increases. Then, an equilibrium allocation which satisfies the law of price in a game $\mathbf{E}^{\kappa}$, is an equilibrium allocation for every market game $\mathbf{E}^{\tau}$ whatever is $\tau \neq \kappa$ (as far as $\tau > \mathbf{0}$).

[13] It is unclear whether the proofs stated by Koutsougeras (2003*b*) applies to other situations than the one without liquidity constraint ($C_i\,(b, q) = \emptyset$). The proofs bears on two facts Koutoutsougeras (2003*b*) specifies and that—as the author notices—cannot be used when at least one endowment constraint is binding. Our analysis shows they do.

In conclusion, the number of trading-posts is not relevant for Nash equilibrium allocations which do not feature wash-sales. Therefore, considering wash-sales as artificial trades—because there is nothing that can be said about what incentives a given agent has to trade wash-sales—is sufficient to conclude there is no gain in assuming more than one post per commodity type. In this case, adding trading-posts increases the strategy space—the (proportional) amounts on all active trading-posts for a given commodity can be modified in proportion for all agents—but all those similar Nash equilibria parametrized by the gross volume of trade on different trading-posts achieve the same allocations.

8 Conclusion

When agents do not support transaction costs, that they can be unable to trade off dispersed prices for themselves is unexpected. In this paper, we have highlighted the importance of agents to be large on the market, and the interest of being not to too large on a single trading-posts. This may be contradicting to an usual idea, which is that within imperfect competitive environment the more agents are able to monopolize a market the more they can turn the price for themselves. This idea is only partially true in the market game setting in which agents should not cannibalize a trading-post. Then, the role of liquidity constraints has been qualified. Contrary to the barter economy, they may at most limit arbitrage and never induce dispersion. The reason lies in the fact that money allows to trade on all markets, so that given moves of others a rational agent will always reduce price dispersion by taking advantage of it in the money market game. Then, this weak influence of liquidity constraints has then been underlined by a convergence result to price uniformity when the number of agents trading a commodity increases, that there are constrained or not. Finally, we have argued for the weak influence of the market structure in the sense that the number of trading-posts is not relevant for equilibrium allocation.

Our exploration of liquidity considerations does not explicitly include the role of *money* liquidity constraints. However, the role of *money* liquidity constraints is not of different nature. According to the analyses pursued within the example, it is expected that the results obtained here for commodity liquidity constraints apply to money liquidity constraints. This idea is partially confirmed by the analysis of a Shapley - Shubik market game with money constraints and transactions costs (Breton and Gobillard, 2006). Yet, this leaves the question of the role of market liquidity open. One may wonder whether increasing the thickness of the posts using short-sales (in the vein of Peck and Shell, 1990) may induce a convergence result when the number of agents is finite, contrary to wash-sales being necessary to the existence of price dispersion. This question is left for future research.

A Best reply properties

Proof of lemma (7)**.** Agent h acts so as to maximize $u\left(x^h\right)$ over strategies $\sigma^h \in S_h$ s.t. (3) holds. The program is the following.

$$\mathcal{P}_\sigma \left| \begin{array}{l} \max\limits_{(\sigma_h)\in S_h} u_h\left(x_h\right) \\ \sum\limits_{i=1}^{L}\sum\limits_{s=1}^{K_i} b_h^{i,s} - \sum\limits_{i=1}^{L}\sum\limits_{s=1}^{K_i} q_h^{i,s}\left(B_h^{i,s}+b_h^{i,s}\right)/\left(Q_h^{i,s}+q_h^{i,s}\right) \leq 0 \\ \sum\limits_{s=1}^{K_i} q_h^{i,s} - e_h^i \leq 0, \forall\ i=1,...,L \\ -q_h^{i,s} \leq 0, \forall\ s=1,...,K_i \text{ and } i=1,...,L \\ -b_h^{i,s} \leq 0, \forall\ s=1,...,K_i \text{ and } i=1,...,L \end{array} \right. \tag{A.1}$$

Form the Lagrangian of $\mathcal{P}_\sigma$:

$$\mathcal{L}_h = u_h\left(x_h\left(\sigma_h\right)\right) + \sum_{i=1}^{L}\left[\begin{array}{l} \gamma_h \sum\limits_{s=1}^{K_i}\left(b_h^{i,s} - q_h^{i,s}p^{i,s}\right) + \lambda_h^i\left(\sum\limits_{s=1}^{K_i} q_h^{i,s} - e_h^i\right) \\ -\sum\limits_{s=1}^{K_i}\left(\alpha_h^{i,s}q_h^{i,s} + \beta_h^{i,s}b_h^{i,s}\right) \end{array}\right]$$

It is easy to show that the qualification constraint is satisfied. So, if σ_h is a solution to $\mathcal{P}_\sigma$ there is a set of multipliers $\left(\gamma_h, \lambda_h^i, \alpha_h^{i,s}, \beta_h^{i,s}\right) \in I\!R^{L+1} \times \prod\limits_{i=1}^{L} I\!R^{2K_i}$ such that, given (1), first order conditions are:

$$\partial u_h/\partial x_h^i = -\gamma_h\left(p^{i,s}\right)^2 Q_h^{i,s}/B_h^{i,s} + \beta_h^{i,s}p^{i,s}\left(B_h^{i,s}+b_h^{i,s}\right)/B_h^{i,s} \tag{A.2}$$

$$\partial u_h/\partial x_h^i = -\gamma_h\left(p^{i,s}\right)^2 Q_h^{i,s}/B_h^{i,s} + \left(\lambda_h^i - \alpha^{i,s}\right)\left(B_h^{i,s}+b_h^{i,s}\right)/B_h^{i,s} \tag{A.3}$$

Let us note that this implies:

$$\beta_h^{i,s}p^{i,s} = \left(\lambda_h^i - \alpha^{i,s}\right) \tag{A.4}$$

By Eq. (A.4) the case $\lambda_h^i = 0$ is obvious ($\beta_h^{i,s} = \lambda_h^i = \alpha^{i,s} = 0$).

Suppose $\lambda_h^i < 0$. There are posts (i,s) such that $q_h^{i,s} > 0$ and then $\alpha^{i,s} = 0$. Note that the constraint is binding on all posts where $q_h^{i,s} > 0$, with $\beta_h^{i,s}p^{i,s} = \lambda_h^i$. We have the same result if $q_h^{i,s} = 0$ and $\alpha_h^{i,s} = 0$ (in which case h does neither increase or decrease $q_h^{i,s}$). Consider now posts (i,r) where $q_h^{i,r} = 0$ and $\alpha^{i,r} < 0$, situation in which we know the endowment constraint is not preventing agent h to increase $q_h^{i,r}$. In that case, agent h is obviously not constrained by $b_h^{i,s} \geq 0$, $\beta_h^{i,s} = 0$ and $\lambda_h^i = \alpha^{i,s}$. **End proof.**

Proof of proposition (8). When $\beta_h^{i,s} > 0$, $b_h^{i,s} = 0$ and the first order conditions can be rewritten as:

$$\begin{array}{ll} \partial u_h / \partial x_h^i = -\gamma_h \left(p^{i,s}\right)^2 Q_h^{i,s} / B_h^{i,s} + \left(\lambda_h^i - \alpha^{i,s}\right) & \\ \text{with} \left| \begin{array}{l} \alpha_h^{i,s} = 0 \text{ if } \sum\limits_{s=1}^{K_i} q_h^{i,s} \leq e_h^i \text{ is binding on post } (i,s) \\ \alpha_h^{i,s} = \lambda_h^i \text{ otherwise} \end{array} \right. & \end{array} \tag{A.5}$$

So, if (i,s)' bidding non-negativity constraint of agent h is binding, agent h is constrained by his initial endowment and the associated cost defined by the Lagrange multiplier takes the same value. When he is not constrained, none of both these constraints is binding on post (i,s), and first order conditions in respect to $b_h^{i,s}$ or $q_h^{i,s}$ are the same. **End proof.**

Let us now denote, for each post (i,s),

$$x_h^{i,s} = \tfrac{1}{K^i} e_h^i - q_h^{i,s} + b_h^{i,s} Q^{i,s} / B^{i,s} \tag{A.6}$$

Therefore, using (1) and (2) $q_h^{i,s}$ can be rewritten as follows:

$$q_h^{i,s} = \left[B^{i,s} / B_h^{i,s}\right] \left[\tfrac{1}{K^i} e_h^i + Q_h^{i,s} b_h^{i,s} / B^{i,s} - x_h^{i,s}\right] \tag{A.7}$$

Lemma 21 *In the Bid - Offer market game an agent $h \in H$ acts so as to solve the following program $\mathcal{P}_x$.*

$$\mathcal{P}_x \left| \begin{array}{l} \max\limits_{x_h^{i,s} \in \prod_{i=1}^{L} I\!R^{K_i}} \quad u_h\left(x_h\right) \\ -\sum_{i=1}^{L} \sum_{s=1}^{K^i} B_h^{i,s} \left(\tfrac{1}{K^i} e_h^i - x_h^{i,s}\right) / \left[Q_h^{i,s} + \tfrac{1}{K^i} e_h^i - x_h^{i,s}\right] \leq 0 \\ \sum\limits_{s=1}^{K_i} q_h^{i,s} - e_h^i \leq 0, \forall\ i = 1, ..., L \\ -q_h^{i,s} \leq 0, \forall\ s = i = 1, ..., K_i \textit{ and } i = 1, ..., L \end{array} \right. \tag{A.8}$$

Proof. By proposition (8), we avoid constraint $b_h^{i,s} \geq 0$. Then, allocation rule (A.6) can be rewritten as

$$-x_h^{i,s} + \tfrac{1}{K^i} e_h^i + Q_h^{i,s} = B_h^{i,s} Q^{i,s} / B^{i,s} \tag{A.9}$$

Using the price definition we have:

$$\begin{aligned} B_h^{i,s} Q^{i,s} / B^{i,s} &= Q_h^{i,s} + q_h^{i,s} - b_h^{i,s} Q^{i,s} / B^{i,s} \\ &= \left[1/p^{i,s}\right] \left[Q_h^{i,s} p^{i,s} + p^{i,s} q_h^{i,s} - b_h^{i,s}\right] \end{aligned} \tag{A.10}$$

and:

$$\begin{aligned}-p^{i,s}q_h^{i,s}+b_h^{i,s} &= Q_h^{i,s}p^{i,s}-B_h^{i,s}\\ &= p^{i,s}\left[x_h^{i,s}-\tfrac{1}{K^i}e_h^i\right]\end{aligned}$$

Now, as

$$p^{i,s}=B_h^{i,s}/\left[-x_h^{i,s}+\tfrac{1}{K^i}e_h^i+Q_h^{i,s}\right]$$

we obtain:

$$-p^{i,s}q_h^{i,s}+b_h^{i,s}=-B_h^{i,s}\left[\tfrac{1}{K^i}e_h^i-x_h^{i,s}\right]/\left[-x_h^{i,s}+\tfrac{1}{K^i}e_h^i+Q_h^{i,s}\right]$$

and as far as $p^{i,s}>0$, the budget constraint can be rewritten

$$bc_h=-\textstyle\sum_{i=1}^{L}\sum_{s=1}^{K^i}B_h^{i,s}\left(\tfrac{1}{K^i}e_h^i-x_h^{i,s}\right)/\left[Q_h^{i,s}+\tfrac{1}{K^i}e_h^i-x_h^{i,s}\right]\leq 0 \tag{A.11}$$

and this completes the proof. **End proof.**

Lemma 22 *The program $\mathcal{P}_x$ is convex.*

Proof. The utility functions are concave, and the subset defined by the group of constraints is convex. First, the budget constraint bc_h is a convex function of $x_h^{i,s}$.

$$\frac{\partial bc_h}{\partial x_h^{i,s}}=\frac{Q_h^{i,s}B_h^{i,s}}{\left[Q_h^{i,s}+\frac{1}{K^i}e_h^i-x_h^{i,s}\right]^2}\text{ and }\frac{\partial^2 bc_h}{\partial\left(x_h^{i,s}\right)^2}=Q_h^{i,s}B_h^{i,s}/\left[Q_h^{i,s}+\tfrac{1}{K^i}e_h^i-x_h^{i,s}\right]^3$$
$$=\left[Q_h^{i,s}/\left(B_h^{i,s}\right)^2\right]\left(B^{i,s}/Q^{i,s}\right)^3$$

implying that bc_h is a convex function and that the allocation set defined by $bc_h\leq 0$ is convex too. Then, $q_h^{i,s}$ is a linear function satisfying $\partial q_h^{i,s}/\partial x_h^{i,s}=-B^{i,s}/B_h^{i,s}<0$. **End proof.**

It is now possible to complete the proof of proposition (6).

Proof of proposition (6). Form the Lagrangian of $\mathcal{P}_x$:

$$\begin{aligned}L_h=u\left(x_h\right)&+\gamma_h\left(\textstyle\sum_{i=1}^{L}\sum_{s=1}^{K^i}B_h^{i,s}\left(\tfrac{1}{K^i}e_h^i-x_h^{i,s}\right)/\left[Q_h^{i,s}+\tfrac{1}{K^i}e_h^i-x_h^{i,s}\right]\right)\\&-\sum_{i=1}^{L}\sum_{s=1}^{K_i}\alpha_h^{i,s}\left[B^{i,s}/B_h^{i,s}\right]\left[\tfrac{1}{K^i}e_h^i+Q_h^{i,s}b_h^{i,s}/B^{i,s}-x_h^{i,s}\right]\\&+\sum_{i=1}^{L}\left[\lambda_h^i\left(\sum_{s=1}^{K_i}\left[B^{i,s}/B_h^{i,s}\right]\left[\tfrac{1}{K^i}e_h^i+Q_h^{i,s}b_h^{i,s}/B^{i,s}-x_h^{i,s}\right]-e_h^i\right)\right]\end{aligned}$$

with the set of multipliers $\left(\gamma_h,\lambda_h^i\alpha_h^{i,s}\right)\in I\!R\times I\!R^L\times\prod_{i=1}^{L}I\!R^{K_i}$. As the program is convex, there is a unique optimum which necessary and sufficient conditions are given by the following system:

$$\begin{array}{lll} \nabla L_h = 0 & bc_h = 0 & \gamma_h \leq 0 \\ \forall\ i = 1, ..., L & \lambda_h^i \leq 0 & \lambda_h^i \sum\limits_{s=1}^{K_i} \left(q_h^{i,s} - e_h^i \right) = 0 \\ \forall\ s = 1, ..., K_i & \alpha_h^{i,s} \leq 0 & \alpha_h^{i,s} q_h^{i,s} = 0 \end{array}$$

Using straightforward manipulations, we obtain:

$$\partial u_h / \partial x_h^i = -\gamma_h \left(p^{i,s} \right)^2 Q_h^{i,s} / B_h^{i,s} + \left(\lambda_h^i - \alpha^{i,s} \right) \left(B_h^{i,s} + b_h^{i,s} \right) / B_h^{i,s}$$

Now, by (A.5) and definition (5) the first order conditions can be rewritten as:

$$\partial u_h / \partial x_h^i = -\gamma_h \left(p^{i,s} \right)^2 Q_h^{i,s} / B_h^{i,s} + \lambda_h^{i,s}$$

This completes the proof, as it is obvious that $\gamma_h = 0$ implies all marginal utilities have to be non-positive. **End proof.**

Proof of proposition (9)**.** We proceed by stating the following lemma. It is a well-known result, which is extended to the multiple trading-posts setup.

Lemma 23 *Consider moves of others $\sigma_{-h} \in S_{-h}$ as given and strictly positive, and a strategy with wash-sales $\sigma_h \in S_h$. The similar strategy without wash-sales $\hat{\sigma}_h \in S_h$ achieves the same outcome and prices.*

Proof. We show that, given moves of others $\sigma_{-h} \in S_{-h}$, the net outcome of h and the price obtained on a given post by a strategy with wash-sales is the same as the one obtained with the similar profile that does not yield wash-sales. Let us focus on a post (i, s). $\triangle x_h^{i,s}$ denotes the net trade of agent h at post (i, s):

$$\triangle x_h^{i,s} = -q_h^{i,s} + b_h^{i,s} Q^{i,s} / B^{i,s} \tag{A.12}$$

which means that $x_h^i = e_h^i + \sum_{s=1}^{K^i} \triangle x_h^{i,s}$. Denote $\hat{\sigma}_h^{i,s} = \left(\hat{b}_h^{i;s}, \hat{q}_h^{i,s} \right)$ the strategy profile without wash-sales. We distinguish situations agent is a net buyer or a net seller.

In case agent h is a net buyer on post (i, s), that is that $\triangle x_h^{i,s} > 0$, let us define the strategy $\left(\hat{b}_h^{i,s}, \hat{q}_h^{i,s} \right)$ on post (i, s) as $\left(\hat{b}_h^{i,s}, \hat{q}_h^{i,s} \right) = \left(b_h^{i,s} - p^{i,s} q_h^{i,s}, 0 \right)$. The prices are the same in both cases if we have

$$\left[B_h^{i,s} + \hat{b}_h^{i,s} \right] / \left[Q_h^{i,s} + \hat{q}_h^{i,s} \right] = \left[B_h^{i,s} + b_h^{i,s} \right] / \left[Q_h^{i,s} + q_h^{i,s} \right]$$

which is equivalent to

$$\left[B_h^{i,s} + b_h^{i,s} - p^{i,s} q_h^{i,s}\right] / Q_h^{i,s} = \left[B_h^{i,s} + b_h^{i,s}\right] / \left[Q_h^{i,s} + q_h^{i,s}\right]$$

This condition is checked if

$$\left(-p^{i,s} q_h^{i,s}\right) \left(Q_h^{i,s} + q_h^{i,s}\right) + \left(B_h^{i,s} + b_h^{i,s}\right) q_h^{i,s} = 0$$

which is obvious given the definition of prices. Thus, the price on (i, s) is the same in both cases and we have:

$$\triangle \hat{x}_h^{i,s} = -\hat{q}_h^{i,s} + \hat{b}_h^{i,s} / p^{i,s} = \left[1/p^{i,s}\right] \left[b_h^{i,s} - p^{i,s} q_h^{i,s}\right] = \triangle x_h^{i,s} \tag{A.13}$$

which means that h's net trade on post (i, s) is unchanged.

In case agent h is a net seller, let us define $\left(\hat{b}_h^{i,s}, \hat{q}_h^{i,s}\right) = \left(0, q_h^{i,s} - b_h^{i,s}/p^{i,s}\right)$. We have:

$$\hat{p}^{i,s} = B_h^{i,s} / \left[Q_h^{i,s} + q_h^{i,s} - b_h^{i,s}/p^{i,s}\right] = p^{i,s} B_h^{i,s} / \left[p^{i,s} \left(Q_h^{i,s} + q_h^{i,s}\right) - b_h^{i,s}\right] = p^{i,s} \tag{A.14}$$

given the definition of $p^{i,s}$. Then, we have

$$\triangle \hat{x}_h^{i,s} = -\hat{q}_h^{i,s} + \hat{b}_h^{i,s} / p^{i,s} = -\left(q_h^{i,s} - b_h^{i,s}/p^{i,s}\right) = \triangle x_h^{i,s} \tag{A.15}$$

Therefore, h's net trade and prices on post (i, s) are the same in both cases.

The last step is to check $\hat{\sigma}_h \in S_h$. Now, the difference in money holdings is clearly the same in both cases, as the net outcome on a post in terms of unit of account is the opposite of value of the net trade $\triangle x_h^{i,s}$. **End Proof.**

By lemma 23, prices and allocations are the same in both cases. Therefore, it is sufficient to show that if a profile σ_h with wash-sales is a best reply the similar profile without wash-sales $\hat{\sigma}_h$ is a best reply too. By proposition 6, it follows (the terms of conditions (14) are identical) that necessary and sufficient conditions for $\hat{\sigma}_h$ to be a best reply are given by proposition (6) and the fact that the strategy profile satisfies $b_h^{i,s} q_h^{i,s} = 0$, $\forall\ i,\ s$. The no wash-sales requirement is not binding to choose the best reply, it only imposes agents not to enter both sides of any post. That is, necessary and sufficient conditions are identical apart from this additional condition. **End Proof.**

B Convergence theorem 19.

The demonstration is an adjustment of the one of theorem 15. Recall post $(i,1)$ is the less expensive trading-post.

Lemma 24 $g^i(\sigma_n) \leq \sup_{r=1,2,...,T_i} \left(\frac{b_h^{i,1}}{B_h^{i,1}} + \frac{q_h^{i,r}}{Q_h^{i,r}} + \frac{b_h^{i,1}}{B_h^{i,1}} \frac{q_h^{i,r}}{Q_h^{i,r}} \right)$, $\forall\, h \in H$

Proof. On posts where agent h is constrained by his commodity endowment, he does not bid for commodity. Now, we know that if agent h offers commodity i on a post (when he does not bid for commodity), he offers this commodity on other more expensive posts (proposition 11). This implies that we have $\lambda_h^{i,1} \geq \lambda_h^{i,r}$ for each post (i,r), and using (15):

$$\left(p^{i,r}\right)^2 Q_h^{i,r}/B_h^{i,r} \leq \left(p^{i,1}\right)^2 Q_h^{i,1}/B_h^{i,1}$$

which can be rewritten, by straightforward manipulations, as:

$$\begin{aligned} \frac{p^{i,r}}{p^{i,1}} &\leq \frac{B_h^{i,r}}{Q_h^{i,r}} \frac{Q^{i,r}}{B^{i,r}} \frac{B^{i,1}}{Q^{i,1}} \frac{B_h^{i,r}}{Q_h^{i,r}} = \frac{Q_h^{i,1}}{Q^{i,1}} \frac{B_h^{i,r}}{B^{i,r}} \left(1 + \frac{b_h^{i,1}}{B_h^{i,1}} + \frac{q_h^{i,r}}{Q_h^{i,r}} + \frac{b_h^{i,1}}{B_h^{i,1}} \frac{q_h^{i,r}}{Q_h^{i,r}}\right) \\ &\leq \left(1 + \frac{b_h^{i,1}}{B_h^{i,1}} + \frac{q_h^{i,r}}{Q_h^{i,r}} + \frac{b_h^{i,1}}{B_h^{i,1}} \frac{q_h^{i,r}}{Q_h^{i,r}}\right) \end{aligned} \tag{B.1}$$

As the same inequality can be stated for each agent, we have:

$$\sup_{r=1,2,...,T_i} \left(\frac{p^{i,r}}{p^{i,1}} - 1 \right) \leq \sup_{r=1,2,...,T_i} \left(\frac{b_h^{i,1}}{B_h^{i,1}} + \frac{q_h^{i,r}}{Q_h^{i,r}} + \frac{b_h^{i,1}}{B_h^{i,1}} \frac{q_h^{i,r}}{Q_h^{i,r}} \right), \forall\, h \in H$$

End Proof.

The following lemma follows.

Lemma 25 *Let $\sigma \in S$ be an equilibrium profile with a non uniform distribution of prices (we must have $T_i > 1$ for some $i = 1, ..., L$). Then:*

(i) Given $\varepsilon > 0$, we have

$$\#A_n^i \geq \left(T_n^i + 1\right) \left(\tfrac{\sqrt{1+\varepsilon}}{\sqrt{1+\varepsilon}-1}\right) \Rightarrow g^i(\sigma_n) \leq \varepsilon$$

(ii) If $\#A_n^i \geq (T_n^i + 1)$, then

$$g^i(\sigma_n) \leq \frac{\left(T_n^i + 1\right)\left[2\#A_n^i - \left(T_n^i + 1\right)\right]}{\left[\#A_n^i - \left(T_n^i + 1\right)\right]^2}$$

Proof. We show the contrapositive of (i). Suppose $g^i(\sigma_n) > \varepsilon$. Define

$$\theta_h^i = \max\left\{\frac{b_h^{i,1}}{B_h^{i,1}}, \left(\frac{q_h^{i,r}}{B_h^{i,r}}\right)_{r=1}^{r=L}\right\}$$

By lemma (24) we know $g_h^i(\sigma_n) \leq 2\theta_h^i + (\theta_h^i)^2\ \forall\ h$, so that $\theta_h^i > -1 + \sqrt{1+\varepsilon} = \eta(\varepsilon)$. It follows that, $\forall\ h \in A_n^i$, either [14]

$$b_h^{i,1}/B_h^{i,1} > \eta(\varepsilon)/[1+\eta(\varepsilon)]$$

or

$$q_h^{i,r}/B_h^{i,r} > \eta(\varepsilon)/[1+\eta(\varepsilon)]$$

for some $r = 1, ..., K^i$. This implies that, summing over agents and active trading-posts, we have:

$$[(1+\eta(\varepsilon))/\eta(\varepsilon)]\left(T^i+1\right) > \#A_n^i \tag{B.2}$$

which states the contrapositive of (i).

The proof of inequality (ii) is a corollary of the proof above, which says that

$$g^i(\sigma_n) > \varepsilon \Rightarrow [(1+\eta(\varepsilon))/\eta(\varepsilon)]\left(T^i+1\right) > \#A_n^i$$

which can be rewritten as

$$g^i(\sigma_n) > \varepsilon \Rightarrow \#A_n^i > \sqrt{1+\varepsilon}\left[\#A_n^i - \left(T^i+1\right)\right]$$

with positive terms on both sides if $\#A_n^i > T^i + 1$ so that we obtain:

$$\left(\#A_n^i\right)^2 > (1+\varepsilon)\left[\left(\#A_n^i\right)^2 + \left(T^i+1\right)^2 - 2\left(\#A_n^i\right)\left(T^i+1\right)\right]$$

and we have:

$$\frac{(T^i+1)\left[2\#A_n^i - (T^i+1)\right]}{\left[\#A_n^i - (T^i+1)\right]^2} > \varepsilon$$

End Proof.

Proof of theorem 19. By the lemma above, if $\#A_n^i/(T_n^i+1) \to \infty$, we have

$$\begin{aligned} g^i(\sigma_n) \leq \varepsilon &\leq \frac{(T_n^i+1)\left[2\#A_n^i - (T_n^i+1)\right]}{\left[\#A_n^i - (T_n^i+1)\right]^2} \\ &\leq \frac{\left[2\#A_n^i/(T_n^i+1) - 1\right]}{\left[\#A_n^i/(T_n^i+1) - 1\right]^2} \to 0 \end{aligned}$$

[14] It is worth mentionning here that is true only because all agents we consider are trading commodity i, so that $\theta_h^i > 0$.

End Proof.

C Proofs of proposition 20

We proceed by stating intermediate results.

Lemma 26 *Let (p, x) be a set of allocations and prices achieved by a strategy profile σ. If σ is a Nash equilibrium for the game $\mathbf{E}^{\mathbf{k}}$, (p, x) is an equilibrium allocation for the game $\mathbf{E}^{\mathbf{t}}$ as far as $\tau \geq \kappa$.*

Proof. Let x be an allocation of the game $\mathbf{E}^{\kappa}$ achieved by the strategy profile $\{(b_h, q_h) \in S_h^{\kappa} : h \in H\}$. Let us consider the market game $\mathbf{E}^{\mathbf{t}}$ defined as the market game $\mathbf{E}^{\kappa}$, with an additional post $(i, K_i + 1)$ (that is to say, $T_i = K_i + 1$, and $T_j = K_j$ for $j \neq i$). Let x be an allocation of the game $\mathbf{E}^{\kappa}$, achieved by the strategy profile $\sigma^{\kappa} = \{\sigma_h^{\kappa} = (b_h, q_h) \in S_h^{\kappa} : h \in H\}$. Let us choose t_i belonging to $]0, 1[$,[15] and define the profile of strategies $\hat{\sigma}^{\tau} = \left\{ \left(\widehat{b}_h, \widehat{q}_h \right) \in S_h^{\tau} : h \in H \right\}$ for the market game $\mathbf{E}^{\tau}$ where, for each agent $h \in H$:

$$\left(\widehat{b}_h^{i,s}, \widehat{q}_h^{i,s} \right) = \begin{cases} t_i \left(b_h^{K_i}, q_h^{K_i} \right), & s = K_i \\ (1 - t_i) \left(b_h^{K_i}, q_h^{K_i} \right), & s = K_i + 1 \\ \left(b_h^{i,s}, q_h^{i,s} \right) & \text{otherwise} \end{cases} \tag{C.1}$$

First of all it is easy to check that prices and allocations are the same in both cases. Then, using proposition (6) it is easy to state that if each individual strategy σ_h^{κ} is optimal in the market game $\mathbf{E}^{\kappa}$, so are profiles $\hat{\sigma}_h^{\tau}$ in the market game $\mathbf{E}^{\tau}$, and vice versa.

Let us focus on an isolated agent $h \in H$, and consider the moves of others σ_{-h} as given. It is sufficient to notice that the prices and ratios on posts (i, K^i) and $(i, K^i + 1)$ in market game $\mathbf{E}^{\tau}$ are the same as what they were on post (i, K^i) in market game $\mathbf{E}^{\kappa}$. Therefore, if σ^{κ} satisfies conditions (14) in $\mathbf{E}^{\kappa}$, these conditions are satisfied in $\mathbf{E}^{\tau}$ when agents play the profile $\hat{\sigma}^{\tau}$. The result can be extended to the general case, by iteration. **End Proof**

Lemma (26) applies to all kinds of equilibria, whether prices across the different posts where a given commodity is exchanged are uniform or not, but

[15] We follow the proof of Koutsougeras (2003*b*), but we add only one post and not a post for each commodity type, so as to ensure the proofs apply when the numbers of posts for each commodity type are not uniform accross the different commodities.

only when the number of trading posts is increased. Next lemma examines situations where the law of one price is satisfied, and show the equivalence whatever is the number of trading-posts.

Lemma 27 *Let (p, x) be a set of allocations and prices which satisfy the law of one price. If (p, x) is achieved by an equilibrium in the market game $\mathbf{E}^{\mathbf{k}}$, it is an equilibrium allocation for the game $\mathbf{E}^{\tau}$ whatever is the vector $\tau > \mathbf{0}$.*

Proof. The proof is similar as the one of lemma (26), which is extended to situations with a smaller number of posts when the law of one price is satisfied. Let x be an allocation of the game $\mathbf{E}^{\kappa}$ achieved by $\sigma^{\kappa} = \{\sigma_h^{\kappa} = (b_h, q_h) \in S_h^{\kappa} : h \in H\}$. Consider the market game $\mathbf{E}^{\tau}$ as the game $\mathbf{E}^{\kappa}$ without the trading post (i, K_i), that is to say $T_i = K_i - 1$ and $T_j = K_j$ if $j \neq i$. Let us define the profile $\hat{\sigma}^{\tau} = \left\{\hat{\sigma}_h^{\tau} = \left(\widehat{b}_h, \widehat{q}_h\right) \in S_h^{\tau} : h \in H\right\}$ for the market game $\mathbf{E}^{\tau}$ as:

$$\left(\widehat{b}_h^{i,s}, \widehat{q}_h^{i,s}\right) = \begin{cases} \left(b_h^{K_i} + b_h^{K_i - 1}, q_h^{K_i} + q_h^{K_i - 1}\right), & s = K_i - 1 \\ \left(b_h^{i,s}, q_h^{i,s}\right) & \text{otherwise} \end{cases} \tag{C.2}$$

for each agent $h \in H$. We focus on the behavior of an agent $h \in H$ and consider the moves σ_{-h} of others as given. Now, if we consider the two posts (i, K^i) and $(i, K^i - 1)$, as $p^{i,K^i} = p^{i,K^i - 1}$ by specification (15) of corollary (11):

$$B_h^{i,K^i} / Q_h^{i,K^i} = B_h^{i,K^{i-1}} / Q_h^{i,K^i - 1} \tag{C.3}$$

so that we have:

$$B_h^{i,K^i} / Q_h^{i,K^i} = B_h^{i,K^{i-1}} / B_h^{i,K^i - 1} = \hat{B}_h^{i,K^{i-1}} / \hat{Q}_h^{i,K^i - 1}$$

with $\hat{B}^{i,r}$ and $\hat{Q}^{i,r}$ the (i, r)'s aggregated quantities over the strategies from $\hat{\sigma}$. Hence, the new price on post $(K^i - 1)$ is obviously identical to the previous ones on posts K^i and $(K^i - 1)$. As a result, if conditions (14) are checked when σ^{κ} is played in market game $\mathbf{E}^{\kappa}$, there are checked when $\hat{\sigma}^{\mathbf{t}}$ is played in market game $\mathbf{E}^{\tau}$. Therefore, if individual moves of σ^{κ} are optimal, so are the one of $\hat{\sigma}^{\tau}$, and vice versa. The result can be extended to the general case by iteration. **End Proof**

Proof of proposition 20. (i) The first statement is a direct corollary of lemma (26), that shows that when $\mathbf{0} < \tau < \kappa$ an equilibrium allocation in market game $\mathbf{E}^{\tau}$ is an equilibrium allocation in market game $\mathbf{E}^{\kappa}$. (ii) An equilibrium allocation in $\mathbf{E}^{\kappa}$ which satisfies the law of one price can be achieved by an equilibrium profile σ^{τ} in any other economy $\mathbf{E}^{\tau}$ which satisfies $0 < \tau \neq \kappa$, such that it is an equilibrium allocation in market game $\mathbf{E}^{\mathbf{1}}$. The rest of the proof follows from the fact that by definition equilibria in market game $\mathbf{E^1}$ satisfy the law of one price. (*iii*) We combine two results. First, in the Bid or Sell market game all equilibria satisfy the law of one price. Then, a set of allocations which satisfies the law of one price and is achieved by an equilibrium σ^τ in $\mathbf{E}^\tau$ with $\mathbf{0} < \tau$ can be achieved by an equilibrium in any market game $\mathbf{E}^\kappa$ as far as $\mathbf{0} < \kappa$, such that it holds for $\tau = \mathbf{1}$. **End proof**

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