Near-optimality of sunspot equilibria in the Uzawa-Lucas model

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Abstract

This paper investigates the social planner's solution in the Uzawa-Lucas model with endogenous labor. In absence of multiple equilibria, the suboptimality of the equilibrium path conjectured by Lucas is confirmed. When indeterminacy occurs, however, we makes use of optimal control relaxed problems to prove the absence of optimal trajectory and the near-optimality of sunspot equilibria. In such cases, the pseudo-optimal solutions are chattering and only sunspot equilibria may mimic these trajectories whose criterion functionals approximate the supremum of the relaxed problem. We find that this property occurs for empirically plausible values of the parameters.

Key words: Externality, Human capital, Indeterminacy, Social planner, Relaxed problems.
JEL classification: C61, C62, E32, E6, H61, O4.

1 Introduction

Uzawa (1965) and Lucas (1988) introduce investment in human capital in the traditional neoclassical growth model to render the transitional dynamics implied by the theory consistent with the long-run properties of the U.S. economy and explain the differences across countries in the economic development.

By incorporating a labor-leisure tradeoff and endogenous labor, Benhabib and Perli (1994) find that multiple equilibria exist for empirically very plausible parameters and explore an alternative explanation to the observed differences, namely indeterminacy of equilibria or equivalently self-fulfilling prophecies, rather than differences in technology or preferences. This model allows for the

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possibility that human capital generates a positive externality in the production of final goods or in the production of human capital itself. The original Lucas model provides some evidence from U.S. data for the 1900-57 period in support of the existence of this externality. As shown by Einarsson and Marquis (1996), a positive externality is also required in order for the model to be compatible with the postwar U.S. data.

Lucas does not solve the model completely (especially, the sufficient conditions are not checked) but conjectures that in presence of productive externalities the economy converges to a suboptimal balanced growth path (BGP) where the speed of the human capital accumulation is lower than the efficient speed. The main concern of this paper is to establish under which conditions Lucas' conjecture proves to be true and compare them to the empirical data. It will be shown in the model with endogenous labor that the conjecture is only validated when indeterminacy does not occur.

However, in case of indeterminacy an optimal BGP path does not exist but a sequence of "chattering" consumption/investment plans converges to a supremum. This supremum is not a feasible trajectory of the original problem but is the optimum of an artificially convexified problem, usually denoted by *generalized* or *relaxed problem.*[1] This problem can be viewed as a limiting case of the original problem since the difference of welfare between the optimal relaxed solution and some feasible paths can be made uniformly as small as we wish.[2] For a given margin of error, we are then able to determine a set of *pseudo-optimal* solutions whose criterion functionals have approximately the same value as the supremum. It is worth noting that for some plausible parameter values, sunspot equilibria are able to mimic the pseudo-optimal trajectories, this is especially the case for the calibrated values used by Lucas. We call this property "near-optimality".

The remainder of this paper is organized as follows. Section 2 describes the model setup with endogenous labor. Section 3 defines both the social planner's program and the associated relaxed problem. A relaxed optimal solution is proved to exist. In section 4, we show that although there is no optimal trajectory for the original program, the optimal relaxed trajectory can be approximated by a sequence of admissible trajectories in the non-relaxed problem. Section 5 challenges Lucas's conjecture on the suboptimality of the (decentralized) equilibrium path and shows that it proves to be false when indeterminacy occurs and for empirically plausible parameter values. Section 6 concludes.

[1] Relaxed systems have been extensively studied, for instance by Gamkrelidze (1965), and more recently by Cesari (1983).

[2] It must be noticed, however, that uniform convergence applies only on compact subsets of $\mathbb{R}_+$.

2 The model

The framework is a slightly modified version of Benhabib and Perli (1994), generalizing the standard models of Uzawa (1965) and Lucas (1988) by integrating a labor-leisure tradeoff.

For simplicity, we investigate a model without physical capital. [3] Consumption goods C are then produced with human capital only, according to the following technology: $C = (uh)^\alpha$, where $u \in [u_{min}, 1] \subset [0,1]$ is the share of human capital allocated to the sector of consumption goods. [4] Unlike the original Uzawa-Lucas model and in absence of conclusive estimates on the degree of returns to scale (see e.g. Basu and Fernald (1997)), no externality is imposed in the production function. [5]

The economy is populated by a unit measure of identical infinitely lived consumers. They produce human capital with labor and a share $1-u$ of human capital, using a constant returns to scale technology:

$$\begin{aligned} \dot{h} &= \delta A[(1-u)h]^\theta L^{1-\theta} & \delta > 0, \quad \theta \in (0,1) \\ A &= \bar{h}^{1-\theta}(1-\bar{u})^{\theta\gamma_1}\bar{L}^{\gamma_2(1-\theta)} & \gamma_1,\ \gamma_2 \geq 0, \end{aligned} \tag{1}$$

where A is a productive externality, and $\bar{u}$, $\bar{h}$ and $\bar{L}$ denote the average economy-wide levels of respectively the share u, human capital h and labor L. In a symmetric equilibrium $u = \bar{u}$, $h = \bar{h}$ and $L = \bar{L}$, the aggregate law of motion of human capital becomes:

$$\dot{h} = \delta h(1-u)^{\theta(1+\gamma_1)} L^{(1-\theta)(1+\gamma_2)}, \tag{2}$$

which obviously induces increasing returns to scale in the production of human capital. It is still unclear in the literature whether human capital exhibits constant marginal returns to scale or not. [6] This is a critical assumption for

[3] As shown in Benhabib and Perli (1994), this assumption is not decisive for indeterminacy but greatly facilitates the derivation of analytic results.

[4] The assumption that $u_{min} > 0$ insures that the minimum level of consumption is always strictly positive. This does not alter the dynamics of the model as long as u_{min} is chosen sufficiently close to zero but drastically simplify the exposition of the different theorems we use in the paper.

[5] It is well known since Benhabib and Farmer (1994) that productive externalities favors indeterminacy.

[6] Analyzing the postwar experience of the OECD, Jones (1995) provides some evidence against the prediction of "scale effects" induced by the endogenous growth models that an increase in research effort should lead to more rapid growth, suggesting the existence of decreasing returns on human capital in the aggregate level in our setup.

endogenous growth in the Lucas model. We rather impose private decreasing returns on human capital but maintain a sustained economic growth by imposing an externality on both human capital and labor. Marginal constant returns on human capital are then preserved at the aggregate level. [7]

The representative consumer, owner of human capital, is endowed with one unit of time ($L \leq 1$) and maximizes:

$$\max_{u,C,L} \int_0^\infty U(C(t), L(t)) e^{-\rho t} dt, \tag{3}$$

with:

$$U(C, L) \equiv \log C - \frac{L^{1+\chi}}{1+\chi},$$

where $\rho > 0$ denotes the discount rate.

The Hamiltonian associated to the problem of the pseudo-social planner is:

$$H(L, u, h, \lambda) = \alpha \log(uh) - \frac{L^{1+\chi}}{1+\chi} + \lambda \left[A[(1-u)h]^\theta L^{1-\theta} \right],$$

and the first-order conditions for an interior solution are:

$$\frac{\partial H}{\partial L}(L, u, h, \lambda) = 0 \iff \frac{\alpha(1-u)}{\theta u} = \lambda \dot{h} \tag{4}$$

$$\frac{\partial H}{\partial u}(L, u, h, \lambda) = 0 \iff \frac{L^{1+\chi}}{1-\theta} = \lambda \dot{h} \tag{5}$$

$$\frac{\partial H}{\partial h}(L, u, h, \lambda) = -\dot{\lambda} - \rho\lambda \iff \frac{\dot{\lambda}}{\lambda} = \rho - \frac{\alpha}{\lambda h} - \theta \frac{\dot{h}}{h}, \tag{6}$$

together with the law of motion of human capital, equation (2), and the transversality condition:

$$\lim_{t \to \infty} \lambda(t) h(t) e^{-\rho t} = 0.$$

A trajectory satisfying conditions (4) to (6) and the transversality condition will be called *an equilibrium*. [8] By contrast, a solution of the social planner's

[7] The empirical plausibility of diminishing returns at the individual level but constant returns at the aggregate level has been argued by Lucas himself in his original paper. However, by contrast with Lucas (1988) where $\theta(1+\gamma_1) = 1$, the assumption of diminishing returns on $1-u$ is supported by Gong et al. (2004)'s estimates. We thus decide not to impose any specific value to the parameter γ_1.

[8] The concavity of the problem respect to the controls and the state variable jointly insures that the second-order conditions hold.

program (see infra) will be said *efficient* or *optimal*.

In appendices 7.1 and 7.2, we prove that the dynamics of model reduces to a single difference equation:

$$\frac{\dot{u}}{u} = \frac{1-u}{1-\phi-\zeta u}\left[\rho + \delta(1-\theta-u)(1-u)^{\zeta-1}\left(\frac{\alpha(1-\theta)}{\theta}\frac{1-u}{u}\right)^{\phi}\right],$$

where $\zeta = \theta(1+\gamma_1)$ and $\phi \equiv \frac{(1-\theta)(1+\gamma_2)}{1+\chi}$, with a unique balanced growth path when the externality is low enough, i.e. $\phi+\zeta < 1$, and generally two equilibria otherwise.

This latter pattern exhibits indeterminacy: either a *local indeterminacy* when a continuum of equilibrium paths appears in the neighborhood of the lower balanced growth path (which is topologically stable), or *global indeterminacy* when both balanced growth paths are locally determinate (that is, in our framework, when they are repealing). Agents may then regularly jump from one BGP to the other, only modifying the allocation of human capital across sector.

In the next two sections, we will assume that the condition for indeterminacy, $\phi + \zeta > 1$, is satisfied.

3 Relaxed and non-relaxed optimization problem

Assume a central planner maximizing the representative consumer's utility (3) subject to the aggregate law of motion of human capital, equation (2), with the control variable restriction:

$$x(t) \equiv (L(t), u(t)) \in \Upsilon \equiv [0,1] \times [u_{min}, 1].$$

Since Mangasarian (1966), it is well know that the necessary conditions are also sufficient for a global maximum if the maximand (here the utility function) and the constraint are both differentiable and jointly concave in the variables (L, u, h) and if the costate $\lambda(t) \geqslant 0$ at any period. The traditional sufficiency condition does not hold here since the law of motion of human capital is no longer jointly concave in L, u and h (the production function being quasi-concave). Arrow and Kurz (1970) provide a generalized sufficiency condition for optimality that can be used in some problems where the traditional concavity assumptions do not hold.[9] It can be shown that this theorem does

[9] For a formal proof of the theorem, see Seierstad and Sydsaester (1977) or Seierstad and Sydsaester (1987), theorems (2.5) and (3.14).

not apply in our setup when indeterminacy occurs. [10]

To solve the program, we then introduce a more general problem in which convex combinations of the initial production set vectors are authorized. More precisely: while in the original problem, for any couple $x \in \Upsilon$ and for a pre-determined stock of capital h, the representative firm can only produce up to $(1-u)^{\zeta} h L^{\phi(1+\chi)}$ units of output, we now extend the production set and consider that the firm is able to produce $\pi(1-u_1)^{\zeta} h L_1^{\phi(1+\chi)} + (1-\pi)(1-u_2)^{\zeta} h L_2^{\phi(1+\chi)}$ units of output with the vector of inputs $\pi x_1 + (1-\pi) x_2$, for any $\pi \in [0,1]$ and any $(x_1, x_2) \in \Upsilon^2$. We then move to a production set satisfying the definition of a convex set and make the optimization problem easier. This method of *relaxed* or *generalized* problem gave raise to an extended literature in the field of mathematics, presented by Cesari (1983) for the most basic results. Once the initial production set has been "convexified" by adding its convex hull to the set of feasible allocations, the relaxed problem of the social planner becomes: [11]

$$\max_{v} \int_0^{\infty} \sum_{i=1}^{2} p_i(t) U(C_i(t), L_i(t)) e^{-\rho t} dt, \tag{7}$$

subject to:

$$\dot{h}(t) = \sum_{i=1}^{2} p_i(t)(1-u_i(t))^{\zeta} h(t) L_i(t)^{\phi(1+\chi)} \tag{8}$$

with the control variable restrictions:

$$v(t) \equiv (x_1(t), x_2(t), p_1(t), p_2(t)) \in V \equiv \Upsilon^2 \times [0,1]^2 \tag{9}$$

$$p_1(t) + p_2(t) = 1. \tag{10}$$

Since the convexity of the control set is so easily obtained for relaxed problems, the standard existence theorems necessarily apply and the following proposition can be shown:

Proposition 1 *There exists an optimal pair $(K^*(t), v^*(t))$ to the optimization problem (7)-(10).*

[10] The maximized Hamiltonian we define further is only piecewise convex but not globally convex when the condition for indeterminacy holds. Derivations are available from the author upon request.

[11] When the objective function is not concave, it is necessary to convexify the control set to prove the existence of a solution in the relaxed maximization problem. In our model, the utility function is already concave: this procedure is not necessary. However, we express the relaxed problem in the more general way to fit the formulation usually adopted by the relevant literature. An alternative formulation consists in maximizing $\int_0^{\infty} U(p_1(t)x_1(t) + p_2(t)x_2(t)) e^{-\rho t} dt$ subject to (8). It can be proved that the optimal solution is a chattering corner solution: the (limit) value of the maximand will then be the same whatever objective function we chose.

Proof. This is an application of the Filippov-Cesari theorem. [12] See Appendix 7.3. ■

While the sufficiency conditions of the standard theorems fail to apply in the non-relaxed problem when indeterminacy occurs, an optimal trajectory emerges from the relaxed problem and can be used as a benchmark for the non-relaxed problem. In the next section, it will be seen in particular that there exists in the original optimization problem a sequence of trajectories whose criterion functional converges to the benchmark's optimal welfare. This convergence will be used to prove the absence of an optimal solution in the non-relaxed problem.

4 Pseudo-optimal trajectories

The following proposition establishes the possible approximation of relaxed trajectories, whether optimal or not, by ordinary trajectories of the initial problem.

Proposition 2 *Let $\{h^*(t), v^*(t)\}$ be an admissible pair for the relaxed optimal control problem. Then, there exists a sequence $\{h_i(t), x_i(t)\}_{i=1}^{\infty}$ of admissible pairs for the initial non-relaxed problem such that the sequence of admissible trajectories $\{h_i(t)\}$ converges uniformly to $h^*(t)$ on compact subsets of $[0, +\infty)$.*

Proof. This is an extension by Carlson (1993), theorem 4.2, of Berkovitz (1974) and Cesari (1983) to the case of infinite-horizon problems. See appendix 7.4 for the application. ■

The proposition above includes the uniform convergence of a sequence of approximate non-relaxed trajectories to the optimal relaxed solution when maximizing on $[0, T] \subset [0, +\infty)$. In other words, the relaxed optimal solution is the limit of a sequence of suboptimal trajectories for any finite interval problem defined on $[0, T]$. It is assuredly possible to enlarge the compact set as much as possible to get a sequence of trajectories whose limiting criterion functional has the same value as the supremum of the relaxed problem (7)-(10) as the upper bound T tends to infinity.

Furthermore, it is worth noting that the set of admissible trajectories for the initial problem can be expressed as a subset of degenerated trajectories in the relaxed problem, with $p_i(t) = 1$ and $p_j(t) = 0$, $i, j = \{1, 2\}$ and $j \neq i$.

[12] For a complete proof of the theorem can be found in Cesari (1983), chapter 9. The simplified version presented in this paper is due to Seierstad and Sydsaester (1987), theorem 2.8.

As a consequence, the supremum of the relaxed problem consists of an upper bound for welfare in the non-relaxed problem. And an optimal non-relaxed trajectory, if any, performs at the very most as good as the optimal relaxed trajectory.

In case of indeterminacy, as shown in appendix 7.2, the former trajectory must exhibit an alternation of periods of full labor effort and periods of zero labor effort. The argument can be proved by reducing it to absurdity: assume the optimal non-relaxed trajectory is an "interior" solution and embodies quantities of labor $L \in (0, 1)$. For optimal values of the state and costate variables, by choosing either $L = 0$ or $L = 1$ we can increase the criterion functional. Then the trajectories with $0 < L < 1$ cannot be optimal.

Can this chattering non-relaxed trajectory constitute the optimal solution of the relaxed problem?

Proposition 3 *The optimal relaxed solution cannot be a degenerated trajectory when indeterminacy occurs.*

Proof. See appendix 7.5. ■

The immediate consequence of propositions 2 and 3 is that there is no optimal trajectory in the original problem since for any admissible trajectory approximating the optimal relaxed solution one may select another trajectory whose criterion functional gets closer to the supremum: actually, the sequence of non-relaxed trajectories never reaches the supremum. The intuition for these findings relies on the welfare improvement properties of the chattering solutions. By switching from periods of zero labor effort to periods of full labor effort, the social planner may manage to mimic more or less faithfully the optimal relaxed trajectory. Since this trajectory is not degenerated, it is clear that a faster labor switching at some periods of time can make the economy closer to the relaxed optimal solution.

However, due to the convergence of the sequence of trajectories, it is also clear that adding more switchings to an already highly chattering trajectory has very few impact on welfare improvement. For an artificially low error ε and a supremum value $\hat{J}$ we can define a set of original trajectories whose criterion functional is included in $[\hat{J} - \varepsilon, \hat{J})$. These trajectories will be said *pseudo-optimal*: their criterion functional have almost the same value as the supremum and they exhibit a similar behavior for capital, consumption and labor.

5 Optimal solution and (in)determinacy

When the externalities are not sufficient to generate indeterminacy, that is when $\phi + \zeta < 1$, an interior optimal solution may emerge, as conjectured by Lucas (1988). In that case, existence and uniqueness of the optimal trajectory in the non-relaxed problem is insured by the Arrow sufficient condition since the *Maximized Hamiltonian* is strictly concave.

The Maximized Hamiltonian $\hat{H}$ is the value of the Hamiltonian once the controls have been replaced by their maximized values, obtained by the first-order conditions (11) and (12):

$$\hat{H}(h,\lambda) = \alpha \left[\ln[u(h)h] + \frac{(1-\phi)(1-u(h))}{\zeta u(h)} \right],$$

where $u(h)$ is implicitly defined by $\Omega \frac{(1-u)^{1-\zeta-\phi}}{u^{1-\phi}} = h$ and $\Omega \equiv \frac{\alpha}{\zeta\lambda\delta} \left(\frac{\zeta}{\alpha\phi(1+\chi)} \right)^{\phi}$.

As shown in appendix 7.3, provided $1 - \phi - u\zeta > 0$ for any $u \in U$, the values of the controls obtained by way of the first order conditions clearly maximize the Hamiltonian for optimal values of h and λ and its second derivative of $\hat{H}$ respect to h

$$\frac{\partial^2 \hat{H}}{\partial h^2}(h,\lambda) = -\frac{\alpha}{uh}\frac{du}{dh} \left[-\frac{1-\phi-\zeta u}{1-u} + \frac{\phi+\zeta u}{\zeta u} \right],$$

is negative if and only if the term in brackets is negative. [13]

It cannot be satisfied for $\phi + \zeta > 1$ where the term in brackets is minimal for $\phi = 0$ and equal to $\frac{u(\zeta-1)}{1-u}$. Thus, a unique and determinate balanced growth path (in the decentralized economy) is necessary for the social planner's optimal solution to be interior. However, Arrow's theorem cannot conclude on the sufficiency of this condition since there exist parameter values for which a unique equilibrium emerges but the maximized Hamiltonian is not concave in h.

As in the original model, the efficient growth rate of human capital is higher than the equilibrium growth rate. This conclusion, as well as Lucas' conjecture, fails drastically in the case of indeterminacy. By contrast, the chattering pattern of the pseudo-optimal trajectories makes stop-and-go policies efficient in human capital accumulation. It must be recalled that the near-optimal behavior for agents consists in switching between periods of zero labor effort with a full allocation of human capital to the sector of consumption goods ($u_2^* = 1$) and periods of full labor effort where the share of human capital allocated to

[13] After straightforward derivations, one gets: $\frac{du}{dh} = -\frac{1}{h}\frac{u(1-u)}{1-\phi-\zeta u} < 0$.

the sector of consumption goods is the solution of $u_1^*(1-u_1^*)^{\zeta-1} = \rho/(\zeta\delta)$.[14]

For the standard parameter values of the Uzawa-Lucas model, that is when $\zeta = 1$ (linearity of $\dot{h}$ respect to $1-u$) and θ tends to unity (low elasticity of $\dot{h}$ respect to labor), the reader can refer to appendices 7.1 and 7.2 to check easily that ϕ tends to zero, u_1 and u_1^* to ρ/δ, u_2 and u_2^* to 1.[15] . Furthermore, since $u_1 < \tilde{u}$, the two BGP are locally determinate. In the case of indeterminacy, that is for $\zeta + \phi > 1$, self-fulfilling expectations can then replicate the chattering behavior of the pseudo-optimal trajectories, consisting of arbitrary fast jumps of the labor effort (switching between zero and full labor effort). By contrast, a smooth stabilization scheme (in the Benhabib-Farmer framework, see for instance Guo and Lansing (1998)) fails to replicate the erratic behavior and is likely to deteriorate the representative agent's welfare by suppressing endogenous fluctuations. As pointed out by Christiano and Harrison (1999) in the case of increasing returns to scale, by bunching hard work agents may jointly increase the average level of consumption and decrease the average level of labor effort. The standard objective of preventing the economy from fluctuations is then likely to deteriorate the so-called *bunching effect*.

Near-optimality of self-fulfilling expectations also means that the sunspot equilibria, like the pseudo-optimal trajectories, can be ranked in terms of welfare, according to the more or less rapidity of the switches from one BGP to the other. In absence of optimal solution, each near-optimal sunspot equilibrium is welfare-dominated by another one closer to the optimal relaxed solution.

6 Conclusion

It has been proved in this paper that the form of the optimal solution in the Uzawa-Lucas model with endogenous labor highly depends on the degree of the human capital externality. When it is not sufficient to generate indeterminacy, the optimal solution is a balanced growth path with a (generally) faster human capital accumulation than the equilibrium one. However, when indeterminacy occurs, there is no Pareto optimal solution. Nevertheless, a continuum of pseudo-optimal trajectories converges to a supremum, solution of a generalized optimization program in which the original production set has been "convexified". These pseudo-optimal trajectories are chattering solutions

[14] For periods of full labor effort, the optimal value of u is computed while maximizing $\alpha \ln(uh^*) - \frac{1}{1+\chi} + \lambda^*\delta(1-u)^\zeta h^*$, for optimal values of h and λ, that is for $\lambda^* h^* = s^* = \alpha/\rho$. For $\zeta > 1$, the equation has two roots but only the lower satisfies the second order condition: $1 - \zeta u_1^* > 0$.

[15] Although L_1 does not tend to unity, the growth rate of human capital converges however to the efficient growth rate since $L_1^{*\phi}$ tends to 1

along which agents alternate periods of full labor effort with periods of zero labor effort and can sometimes be replicated by sunspot equilibria for specific (but realistic) values of the parameters.

Lucas' conjecture that for any initial configuration $h(0)$ of human capital, the optimal solution path will converge to some point on an efficient balanced growth path is not generically validated in our setup.[16] But more important, in case of pseudo-optimal chattering solutions, endogenous fluctuations are likely to perform better than any stabilization policy. The role of economic policy could even be to amplify such fluctuations so that the two balanced growth paths coincide with the optimal corner solutions, adequate sunspots helping the economy to switch from one corner to the other. This property of near-optimality of sunspot equilibria sounds pretty singular within the standard literature.

For simplicity, results are derived in a model without physical capital but could be generalized in a model integrating such a component (see Benhabib and Perli (1994) for the study of the decentralized economy). We plan to pursue this project in the near future.

[16] It is not validated with certainty when indeterminacy occurs and may not be validated also when the equilibrium is fully determinate.

7 Appendices

7.1 *Global dynamics*

Let $s \equiv \lambda h$. Rearranging equations (4) and (6):

$$\frac{\dot{s}}{s} = \frac{\dot{\lambda}}{\lambda} + \frac{\dot{h}}{h} = \rho - \frac{\alpha}{s} + (1-\theta)\frac{\dot{h}}{h},$$
$$s = \alpha \frac{1-u}{\theta u}\left(\frac{\dot{h}}{h}\right)^{-1}.$$

Substituting equation (2) into the previous equation and differentiating it:

$$\frac{\dot{s}}{s} = \frac{\zeta u}{1-u}\frac{\dot{u}}{u} + (1-\phi)(1+\chi)\frac{\dot{L}}{L},$$

where $\zeta = \theta(1+\gamma_1)$ and $\phi \equiv \frac{(1-\theta)(1+\gamma_2)}{1+\chi}$.

Equalizing equations (4) and (5), one gets:

$$L = \left(\frac{\alpha(1-\theta)}{\theta}\frac{1-u}{u}\right)^{\frac{1}{1+\chi}}.$$

Finally, the reduced form of the dynamics (4)-(6) is obtained by combination of these four equations:

$$\frac{\dot{u}}{u} = \frac{1-u}{1-\phi-\zeta u}\frac{\dot{s}}{s}$$
$$= \frac{1-u}{1-\phi-\zeta u}\left[\rho + \delta(1-\theta-u)(1-u)^{\zeta-1}\left(\frac{\alpha(1-\theta)}{\theta}\frac{1-u}{u}\right)^{\phi}\right].$$

7.2 *Balanced growth paths*

It has been proved in 7.1 that:

$$\frac{\dot{u}}{u} \equiv P_1(u).P_2(u),$$

where $P_1(u) \equiv \frac{1-u}{1-\phi-\zeta u}$ is positive on $(\tilde{u}, 1]$ and negative otherwise, with $\tilde{u} \equiv \frac{1-\phi}{\zeta}$.

After noticing that:

$$\begin{aligned} P_2'(u) &= \frac{\delta}{u(1-u)} \left(\frac{1-\theta}{\theta}\frac{1-u}{u}\right)^{\phi} P_3(u) \\ P_3(u) &\equiv \zeta u^2 - [1-\phi+(1-\theta)(\zeta-1)]u - (1-\theta)\phi, \end{aligned}$$

where P_3 is a second-order polynomial that tends to $+\infty$ as u tend to $\pm\infty$, with $P_3(0) = -(1-\theta)\phi < 0$ and $P_3(1) = \theta(\phi+\zeta-1)$, two cases must be distinguished.

First, when $\phi+\zeta < 1$, $P_1(u)$ is always negative and $P_3(u)$ has no root between zero and 1. P_2 is then decreasing on $[0,1]$, with $\lim_{u\to 0} P_2(u) = +\infty$ and $\lim_{u\to 1} P_2(u) = -\infty$. There is a unique balanced growth path.

Second, when $\phi+\zeta > 1$, there is exactly one root between zero and 1, that is

$$\bar{u} = \frac{1-\phi+(\zeta-1)(1-\theta)+\sqrt{[1-\phi+(\zeta-1)(1-\theta)]^2+4\phi\zeta(1-\theta)}}{2\zeta}.$$

P_2 is then decreasing on $[0,\bar{u})$ and increasing on $(\bar{u},1]$. For $\phi < 1+(\zeta-1)(1-\theta)$:

$$\begin{aligned} \bar{u} &= \frac{1-\phi+(\zeta-1)(1-\theta)}{2\zeta} \\ &\quad + \frac{[1-\phi-(\zeta-1)(1-\theta)]\sqrt{1+(\phi+\zeta-1)/[1-\phi-(\zeta-1)(1-\theta)]^2}}{2\zeta} \\ &> \frac{1-\phi}{\zeta} = \tilde{u}. \end{aligned}$$

To the contrary, for $\phi \geq 1+(\zeta-1)(1-\theta)$ and $\phi+\zeta > 1$, it must be notice that $\phi(1-\theta) > 1-\phi$ and:

$$\begin{aligned} \bar{u} &= \frac{1-\phi+(\zeta-1)(1-\theta)}{2\zeta} \\ &\quad + \frac{[(\zeta-1)(1-\theta)-1+\phi]\sqrt{1+(\phi+\zeta-1)/[1-\phi-(\zeta-1)(1-\theta)]^2}}{2\zeta} \\ &> \frac{(\zeta-1)(1-\theta)}{\zeta} > \frac{\phi(1-\theta)}{\zeta} > \frac{1-\phi}{\zeta} = \tilde{u}. \end{aligned}$$

Thus, for $\phi + \zeta > 1$, since $P_1(u)$ cannot be equal to zero, there can be up to two equilibria:

- there is no equilibrium when $P_2(\bar{u}) > 0$,
- there is one equilibrium exactly when $P_2(\bar{u}) = 0$ and this equilibrium is $u^* = \bar{u}$,
- there are two equilibria when $P_2(\bar{u}) < 0$, namely u_1 and u_2, such that $u_1 < \bar{u} < u_2$.

Assuming the existence of two equilibria, we can derive easily the sign of $\dot{u}/u$. For $\tilde{u} < u_1$:

u	u_{min}		$\tilde{u}$		u_1		$\bar{u}$		u_2		1
$P_2(u)$		$+$		$+$	0	$-$		$-$	0	$+$	
$P_1(u)$		$-$	‖	$+$		$+$		$+$		$+$	
$\dot{u}/u$		$-$	‖	$+$	0	$-$		$-$	0	$+$	

For any $u(0) \in (\tilde{u}, u_2)$, the dynamics converges to u_1. Thus, the lower equilibrium is indeterminate. To the contrary, the upper equilibrium is locally determinate.

For $\tilde{u} > u_1$:

u	u_{min}		u_1		$\tilde{u}$		$\bar{u}$		u_2		1
$P_2(u)$		$+$	0	$-$		$-$		$-$	0	$+$	
$P_1(u)$		$-$		$-$	‖	$+$		$+$		$+$	
$\dot{u}/u$		$-$	0	$+$	‖	$-$		$-$	0	$+$	

The transversality condition is satisfied only when the economy lies on a balanced growth path. The two equilibria are locally determinate.

7.3 Hessian matrix of $\tilde{H}(L, u, h^*, \lambda^*)$

The Hamiltonian associated to the social planner's program is:

$$\tilde{H}(L, u, h, \lambda) = \alpha \log(uh) - \frac{L^{1+\chi}}{1+\chi} + \lambda \left[(1-u)^{\zeta} h L^{\phi(1+\chi)} \right],$$

Using the first-order conditions of the social planner's program

$$\frac{\partial \tilde{H}}{\partial L}(L,u,h,\lambda)=0 \Longleftrightarrow \frac{\alpha(1-u)}{\zeta u} = \lambda \dot{h} \tag{11}$$

$$\frac{\partial \tilde{H}}{\partial u}(L,u,h,\lambda)=0 \Longleftrightarrow \frac{L^{1+\chi}}{\phi(1+\chi)} = \lambda \dot{h} \tag{12}$$

$$\frac{\partial \tilde{H}}{\partial h}(L,u,h,\lambda)=-\dot{\lambda}-\rho\lambda \Longleftrightarrow \frac{\dot{\lambda}}{\lambda} = \rho - \frac{\alpha}{\lambda h} - \frac{\dot{h}}{h}, \tag{13}$$

and after some algebra, the Hessian matrix H^e of $\tilde{H}(L, u, h^*, \lambda^*)$ can be written as follows: [17]

$$H^e = \begin{pmatrix} -\frac{\zeta L^{1+\chi}(1-\zeta u)}{\phi(1+\chi)u(1-u)^2} & -\frac{\zeta L^{\chi}}{1-u} \\ -\frac{\zeta L^{\chi}}{1-u} & (\phi-1)(1+\chi)L^{\chi-1} \end{pmatrix}.$$

The determinant of the matrix is:

$$Det(H^e) = \frac{\zeta L^{2\chi}}{\phi u(1-u)^2}(1-\phi-\zeta u).$$

It is positive if and only if:

$$1-\phi-\zeta u > 0 \quad \text{for any } u \in U \subset [0,1]. \tag{14}$$

Under condition (14), the trace of the Hessian matrix is strictly negative: both eigenvalues have negative real part. For optimal values of h and λ, the Hamiltonian $\tilde{H}(L, u, h^*, \lambda^*)$ is then jointly concave in $(L, u) \in \Upsilon$. A sufficient condition is that $1 - \phi - \zeta > 0$: in that case, the BGP is unique and then globally determinate according to appendix 7.2. However, it is worth noting that indeterminacy does not rule out this condition.

When condition (14) is not satisfied in the neighborhood of the BGP, one of the two controls must have corner solutions. Provided $h > 0$ and $L > 0$, $\partial\tilde{H}/\partial u = 0$ has a solution between zero and 1 for u and $\partial^2\tilde{H}/\partial u^2 < 0$: the optimal value of u must be interior. Thus, the value of L maximizing the Hamiltonian is either zero or 1.

7.4 Proof of proposition 1

According to the Filippov-Cesari theorem, there exists an optimal pair $(h^*(t), v^*(t))$ to the optimization problem (7)-(10) provided for all $t \in \mathbb{R}_+$ and all admissible pairs $(h(t), v(t))$:

[17] The Hessian matrix is computed for optimal values of h and λ, respectively h^* and λ^*.

i. there exists an admissible pair $(h(t), v(t))$,
ii. for each (h,t) the set $N(h, V, t) \in \mathbb{R}^2$ defined by

$$N(h, V, t) = \left\{ \left(\sum_{i=1}^{2} p_i(t)U(C_i(t), L_i(t))e^{-\rho t} + \eta, g(h, v, t) \right) : \eta \geq 0, v \in V \right\}$$

and

$$g(t, h, u, L) \equiv \sum_{i=1}^{2} p_i(t)[u_i(t)K(t)]^{\alpha} L_i(t)^{\beta} - \tau p_i(t)u_i(t)^{\theta} K(t) - p_i(t)C_i(t)$$

is convex,
iii. Υ is closed and bounded,
iv. there exist piecewise continuous functions m and j such that

$$\left| \sum_{i=1}^{2} p_i(t)U(C_i(t), L_i(t))e^{-\rho t} \right| \leq m(t)|h| + j(t),$$

for all (h, t), $v \in V$.

Conditions i. and ii. are straightforwardly satisfied: the relaxed problem has been built so as to specifically satisfy condition ii. Condition iii. is defined by assumption: $\Upsilon = [0, 1]^2$.

Since $f : [h(0), +\infty) \times V \times \mathbb{R}_+ \to \mathbb{R}$ defined by

$$f(h, v, t) = \sum_{i=1}^{2} p_i U(C_i(t), L_i(t))e^{-\rho t}$$

is a continuously differentiable mapping and the derivative respect to h is continuously decreasing on $h \in (h(0), +\infty)$, bounded above by $f'_h(h(0), v, t)$ and below by zero, then f is a Lipschitz function and necessarily satisfies condition iv.

7.5 Proof of proposition 2

According to Carlson (1993), there exists a sequence $\{K_i(t), x_i(t)\}_{i=1}^{\infty}$ of admissible pairs for the initial non-relaxed problem such that the sequence of admissible trajectories $\{h_i(t)\}$ converges uniformly to $h^*(t)$ on compact subsets of $[0, +\infty)$ provided:

i. Υ is closed and bounded,
ii. the set given by

$$M = \{(t, h, x) : (t, h) \in [0, \infty) \times [h(0), +\infty), x \in \Upsilon\}$$

is closed,

iii. $F = (g, f) : M \to \mathbb{R}^2$ be a given continuous vector-valued function, and let m be a piecewise continuous function from $[0, +\infty)$ into $\mathbb{R}$ such that

$$|F(t, h, x) - F(t, h', x)| < m(t)|h - h'|$$

holds for almost all $t \geq 0$, $(t, h, x) \in M$ and $(t, h', x) \in M$.

Condition i. has already been proved.

The closure of $[0, \infty)$, $[h(0), +\infty)$ and Υ implies condition ii.

The argument to prove condition iii. is the same as for condition iv. above.

7.6 Proof of proposition 3

It has been proved that for $\phi + \zeta > 1$, the optimal choice for labor must be a corner solution: either $L = 0$ (and $u = 1$) or $L = 1$ (and $u \in [u_m in, 1)$). It must be noticed however that for any $u \in [u_m in, 1]$, $U(h, 0) > U(uh, 1)$ it cannot be optimal to choose $L = 1$ at every period. It is clear that choosing $L = 0$ at every period is not optimal either: selecting $L = 1$ at a given period $\breve{t} \in [0, +\infty)$ does not decrease the value of the objective function but permanently increases the stock of human capital then the intertemporal utility when agents choose $L = 0$ for any $t \in (\breve{t}, +\infty)$. The optimal solution, if any, must alternate periods of zero labor effort and periods of full labor effort.

In the relaxed problem, assume two different trajectories on $[t_0, t_2] \subset [0, +\infty)$ and let t_1 be the real number such that $t_1 - t_0 = t_2 - t_1$. The first trajectory is such that that the strategy $L_1(t) = 0$ (resp. $L_2(t) = 0$) is chosen by the social planner as a degenerated trajectory for any $t \in [t_0, t_1)$ (resp. $t \in [t_1, t_2]$). In other words:

$$p_1(t) = \begin{cases} 1 & \text{for } \ t \in [t_0, t_1) \\ 0 & \text{for } \ t \in [t_1, t_2]. \end{cases}$$

The associated optimal choice for u is: [18]

$$u(t) = \begin{cases} u_1(t) = 1 & \text{for } \ t \in [t_0, t_1) \\ u_2(t) = \tilde{u} & \text{for } \ t \in [t_1, t_2]. \end{cases}$$

[18] When $L = 1$, the problem collapses to maximize $\alpha \ln(uh)$ such that $\dot{h} = \delta(1-u)^{\zeta} h$ or equivalently to maximize $\alpha \ln \left(uh(t_0) e^{\delta(1-u)^{\zeta}} \right)$. It is easy to show that optimal u is unique at every period. We call this value $\tilde{u}$.

This means that h increases only on $[t_1, t_2]$ at rate $\delta(1-\tilde{u})^{\zeta} \equiv \omega$.

To the contrary, select a mixed strategy with $u_1(t) = 1$ and $u_2(t) = \tilde{u}$ for any $t \in [t_0, t_2]$, and:

$$p_1(t) = \begin{cases} p_1 \in (0,1) & \text{for } \; t \in [t_0, t_1) \\ p_1' \in (0,1) & \text{for } \; t \in [t_1, t_2]. \end{cases}$$

In this strategy, h increases at rate $(1-p_1)\omega$ on $[t_0, t_1)$ and at rate $(1-p_1')\omega$ on $[t_1, t_2]$.

For any $\varepsilon \in [0, t_1 - t_0)$, the instantaneous utility of periods $t_0 + \varepsilon$ and $t_1 + \varepsilon$ is for the degenerated strategy:

$$U_d(t_0+\varepsilon, t_1+\varepsilon) = U(h(t_0), 0)e^{-\rho(t_0+\varepsilon)} + U(\tilde{u}h(t_1+\varepsilon), 1)e^{-\rho(t_1+\varepsilon)},$$

and for the mixed strategy:

$$\begin{aligned} U_m(t_0+\varepsilon \;,\; t_1+\varepsilon) = \\ & [p_1 U(h(t_0+\varepsilon), 0) + (1-p_1)U(u(t_0+\varepsilon)h(t_0+\varepsilon), 1)]\, e^{-\rho(t_0+\varepsilon)} \\ & + [p_1' U(h(t_1+\varepsilon), 0) + (1-p_1')U(u(t_1+\varepsilon)h(t_1+\varepsilon), 1)]\, e^{-\rho(t_1+\varepsilon)}. \end{aligned}$$

By setting arbitrarily $1-p_1 = (1-p_1')e^{-\rho(t_1-t_0)}$ and after some straightforward derivations, it can be proved that:

$$U_m(t_0+\varepsilon, t_1+\varepsilon) = U_d(t_0+\varepsilon, t_1+\varepsilon) + \alpha\omega(1-p_1)(t_1-t_0)e^{t_1+\epsilon},$$

which is strictly greater than $U_d(t_0+\varepsilon, t_1+\varepsilon)$ for $p_1 \in (0,1)$.

Since the inequality is true for any $\varepsilon \in [t_0, t_1)$, the mixed strategy makes the agents better-off and the social planner has no incentive to choose any degenerated strategy.

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