A New Keynesian Model with Heterogeneous Expectations*

William A. Branch
University of California, Irvine

Bruce McGough
Oregon State University

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Abstract

This paper introduces heterogeneous expectations into a New Keynesian model. Our primary theoretical contribution is to provide an aggregation result for a micro-founded model with nominal rigidities and heterogeneous expectations. We show that equilibrium output and inflation are solutions to a heterogeneous expectations version of the New Keynesian reduced form relations. With heterogeneous agents, the Phillips curve depends on current expectations of both contemporaneous and future endogenous state variables. As an illustration of the potential importance of heterogeneity, we show that incorporating heterogeneous expectations into the New Keynesian model may significantly alter its determinacy properties: models that are determinate under the assumption of rationality may possess multiple equilibria in the presence of expectations heterogeneity, even for small departures from rationality.

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1 Introduction

During the past decade the New Keynesian monetary model has become a benchmark laboratory for the analysis of monetary policy. Despite a literature that has

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coordinated on this model, there is a surprisingly diverse set of policy advice: some advocate simple, implementable policy rules (McCallum and Nelson (1999), Schmitt-Grohe and Uribe (2005)); optimal state-contingent monetary policy (Giannoni and Woodford (2002), Woodford (2003), Benigno and Woodford (2005)); or, inflation targeting or inflation forecast targeting (Svensson (1999), Svensson and Woodford (2003), Svensson and Williams (2005)). One theme emerging from this literature is that policy should account for its effect on private-sector expectations of future policy actions. Indeed, Woodford (2006) emphasizes the role policy inertia plays in influencing expectations. Even in the presence of non-rational agents, it is still advised that policy be set while remaining cognizant of the feedback onto beliefs, though often with the added condition that the associated rational expectations equilibrium (REE) is stable under learning (Bullard and Mitra (2002)).

The emergence of this theme stems from models that assume a representative agent structure and homogeneous private-sector expectations.[1] This assumption is widespread despite increasing evidence that agents have heterogeneous expectations. For example, Mankiw, Reis and Wolfers (2003) document disagreement among both consumers and professional economists in survey data on inflation expectations. Carroll (2003) shows that information slowly diffuses through the economy. Branch (2004, 2005) finds that survey data on inflation expectations are consistent with a dynamic discrete choice between statistical predictor functions. The dynamic effect of expectations heterogeneity, though, has not yet been studied in the context of micro-founded monetary models. This paper introduces heterogeneous expectations into a New Keynesian framework and studies its implications for the dynamic equilibrium properties of the economy.

This paper's principal contribution is it provides a theory of heterogeneous expectations that generalizes the literature on monetary policy in New Keynesian models to a framework with an empirically realistic model of expectation formation. We derive a heterogeneous expectations version of the New Keynesian model employing micro-foundations along the lines of Woodford (2003).[2] The aggregation of heterogeneous expectations into a reduced form from micro-foundations is non-trivial, and we make explicit the conditions under which aggregation obtains.

Surprisingly, our formulation of heterogeneous expectations leads to a reduced form similar to the standard IS and AS relations except that conditional expectations are replaced by a general expectations operator that is a convex combination of heterogeneous expectations. Also, depending on the behavior of boundedly rational

[1]Heterogeneous expectations between the policymakers and the private sector have been analyzed by Honkapohja and Mitra (2006) and Sargent (1999).

[2]Gali, Lopez-Sallido, and Valles (2004) develop a model with rational and myopic agents. The heterogeneity is in consumption-saving behavior rather than expectations. In our model, all agents solve the same problem, have the same utility function, and face the same constraints. They differ, however, in how they evaluate their utility flows, which may imply different individual wealth levels.

agents, the heterogeneous expectations aggregate supply equation conditions on current expectations of both future *and* contemporaneous endogenous state variables. These differences imply that the incorporation of heterogeneous expectations into a New Keynesian model has strong implications for the equilibrium dynamics. Indeed, by way of an example, we show that if expectations are a weighted average of rational and adaptive expectations, then even a small degree of heterogeneity can render a determinate model indeterminate.

The primary theoretical result of this paper demonstrates that a micro-founded New Keynesian model with nominal rigidities and heterogeneous expectations aggregates into a tractable reduced form model that can be used to study the conduct of monetary policy. We show that the dynamic properties of New Keynesian models are sensitive to the distribution of heterogeneity. Our application to the design of monetary policy, and its impact on the equilibrium determinacy properties of the model, provides a simple illustration of the importance of heterogeneous expectations.

That deviations from the canonical New Keynesian model can alter the equilibrium properties of the economy has been documented elsewhere. For example, in a New Keynesian model with capital, (local) indeterminacy may result (Carlstrom and Fuerst (2005)), or possibly global multiple equilibria even when the model is locally determinate (Benhabib and Eusepi (2005)). If nominal interest rates face a zero lower bound, Benhabib, Schmitt-Grohe, and Uribe (2001, 2002) show that monetary policy set to achieve a locally determinate steady state may have multiple bounded equilibria and may even converge to an indeterminate steady state. Bullard and Mitra (2002), Evans and Honkapohja (2003a,b), and Preston (2006) show that, under adaptive learning, the unique REE may not be stable for some policy rules.

This paper also relates to a growing interest by economists in turning to heterogeneous agent economies in order to bring macroeconomic models into closer contact with observed empirical regularities: see, for example, Aiyagari (1994, 1995), Constantinides and Duffie (1996), Krebs (2003), Krusell and Smith (1998) and Krueger and Lustig (2006). In these models, agents face uninsurable idiosyncratic risk but all have rational expectations. In our framework, agents are able to share risk efficiently, but they hold heterogeneous expectations.

Our approach generalizes Woodford (2003) to incorporate heterogeneous expectations in such a manner that the principal distinction in the reduced form IS and AS relations between approaches is the specification of the expectations operator. This incorporation does require some care since the usual New Keynesian model already has heterogeneity due to the Calvo pricing mechanism. To identify individual agents' consumption index with aggregate income, the standard treatment assumes complete asset markets, which allow agents to hedge the risks associated with the Calvo lottery. However, complete financial markets and heterogeneous expectations (or asymmetric information) are often thought to be logically inconsistent because the prices in these markets should reveal the relevant information. To avoid this (potential) complication, we assume the existence of a benevolent insurance planner, who takes insurance premiums optimally from each type of consumer and then returns the proceeds to the agents, in order to provide agents the means to hedge Calvo related risk. This highly stylized setting is designed to remain as close as possible, in reduced form, to the standard model, by preserving some semblance of a representative agent. The model in this paper, then, should be seen as a lower bound on the implications of heterogeneity.[3]

We also provide a concrete example – to illustrate potential applications – in which a proportion of agents are rational and the remaining form their expectations adaptively. We find that the impact of increasing the proportion of adaptive agents depends critically on how the adaptive predictor weighs past data. If past data is discounted, then heterogeneity may be stabilizing in the sense that policy rules that result in indeterminacy under full rationality may yield determinacy when even a small proportion of agents are adaptive. On the other hand, if agents place greater weight on past data, then heterogeneity may be destabilizing in an analogous sense: policy rules yielding determinacy under rational expectations may produce indeterminacy if even a small proportion of agents are adaptive.

The heterogeneous expectations framework facilitates a generalization of the manner in which policy affects private-sector expectations. Importantly, monetary policy influences expectations of current and future policy decisions. The conduct of sound policy depends on the distribution of heterogeneity since heterogeneous expectations can alter the way in which expectations propagate shocks. This reasoning ties in with policymakers' frequently stated priority to prevent inflation expectations from becoming unhinged (e.g. Bernanke (2004)). In the current framework, how aggressive monetary policy needs to be in responding to inflation depends on the distribution of heterogeneity and how strongly boundedly rational agents respond to past data. This channel is missing from the standard R.E. model.

One advantage to the approach presented here is that the reduced form nests both the standard model and a heterogeneous expectations version, where the heterogeneity is in line with documented survey evidence. Our approach is deliberately general in order to illustrate the flexibility and because the actual distribution of heterogeneity is an open empirical question. The purpose of this paper is to present a coherent theory and to illustrate its potential importance. The extent to which a heterogeneous expectations model can account for historical time-series is a topic of current research and beyond the scope of the present paper.

This paper is organized as follows. Section 2 generalizes the New Keynesian model to include heterogeneous expectations. Section 3 presents an example with an

[3]We envision that a more realistic modeling of the insurance market will have distinct implications. We leave such an approach to future work.

explicit specification of heterogeneity to facilitate analysis of the model's equilibrium properties. Section 4 concludes.

2 Foundations with Heterogeneous Expectations

This section develops a version of the New Keynesian model extended to include two types of agents who are identical except with respect to the method used to form expectations. We specify the model to remain as close as possible to the standard model with rational expectations, and still allow for aggregation across heterogeneous agents. Our primary finding is an aggregation result: we derive log-linearized IS and AS relations that depend on a convex combination of heterogeneous expectations operators. To ease exposition, we assume a purely forward-looking specification of the model, though it would be straightforward to extend our methodology to a version of the model that incorporates habit persistence and partial price indexation (e.g. Christiano, Eichenbaum, and Evans (2005)).

2.1 Heterogeneous Expectations

Following Woodford (2003) there is a continuum of private agents indexed by $i \in [0, 1]$. These agents are yeoman farmers with linear production functions $Y^i = N^i$, where agent i produces good i by supplying labor N^i. Agents maximize the expected value of discounted utility flow, but instead of assuming that agents form expectations rationally, we allow for more general expectations operators. We assume that a proportion α of agents forecast future variables using the expectations operator E^1 and the remaining agents use E^2, and for simplicity, we assume an agent producing good $i \in [0, \alpha]$ uses E^1 and an agent producing good $j \in (\alpha, 1]$ uses E^2. This paper develops a framework in which equilibrium inflation and output depend on a heterogeneous expectations operator $\hat{E}_t$ that is a linear combination of $E^1, E^2 : \hat{E} = \alpha E^1 + (1 - \alpha)E^2$.

Our view of expectation formation is influenced, in part, by the adaptive learning literature (e.g. Marcet and Sargent (1989), Evans and Honkapohja (2001)). In this literature, fully rational expectations are replaced by linear forecasting rules whose parameters are updated by recursive least squares. We also envision agents engaged in economic forecasting while recognizing there may exist heterogeneity in forecasting rules. Some examples of heterogeneity consistent with our framework include the following: some agents may be rational while others adaptive, as evidence was presented by Branch (2004); agents may have different information sets (e.g. Mankiw, Reis, and Wolfers (2003)); or, they may use structurally different learning rules as in Honkapohja and Mitra (2005). Our goal is to extend this notion of agents as forecasters to the agents' primitive problem and to characterize a set of admissible beliefs that facilitates aggregation into heterogeneous expectations IS and AS relations.

Our characterization of the types of admissible beliefs arises through several assumptions we make regarding E_t^τ, the (subjective) expectations operator. These assumptions are intended to make the model analytically tractable while insisting upon some notion of reasonable agent behavior. We assume

A1. If x is a variable forecasted by agents and has steady state $\bar{x}$ then $E^1\bar{x} = E^2\bar{x} = \bar{x}$.

A2. If x, y, $x + y$ and αx are variables forecasted by agents then $E_t^\tau(x + y) = E_t^\tau(x) + E_t^\tau(y)$ and $E_t^\tau(\alpha x) = \alpha E_t^\tau(x)$.

A3. If for all k, x_{t+k} and $\sum_k \beta^{t+k} x_{t+k}$ are forecasted by agents then

$$E_t^\tau\left(\sum_k \beta^{t+k} x_{t+k}\right) = \sum_k \beta^{t+k} E_t^\tau(x_{t+k}).$$

A4. E_t^τ satisfies the law of iterated expectations (L.I.E.): If x is a variable forecasted by agents at time t and time $t + k$ then $E_t^\tau \circ E_{t+k}^\tau(x) = E_t^\tau(x)$.

A5. If x is a variable forecasted by agents at time t and time $t+k$ then $E_t^\tau E_{t+k}^{\tau'}(x_{t+k}) = E_t^\tau x_{t+k}$, $\tau' \neq \tau$.

These assumptions serve two purposes: they impose logical consistency on the expectations operator and aid in aggregation. The following brief discussion of each assumption is useful. Assumption A1 requires some continuity in beliefs in the sense that, in a steady state, agents' beliefs will coincide. Assumptions A2 and A3 require expectations to possess some linearity properties. Essentially, linear expectations require agents to incorporate some economic structure into their forecasting model rather than, say, mechanically applying a lag operator to every random variable.

Assumptions A4 and A5 restrict agents' expectations so that they satisfy the law of iterated expectations at both an individual and aggregate level. The L.I.E. at the individual level is a reasonable and intuitive assumption: agents should not expect to systematically change their expectations. Assumption A5 is more subtle and is necessary so that aggregate expectations satisfy the law of iterated expectations. The assumption of L.I.E. at the aggregate level is not without consequence, however: A5 requires that an agent of type τ's expectation of the future expectations of agents of type τ' is τ's expectation of that future variable. This assumes away higher-order beliefs of boundedly rational agents. There is a wide literature that studies the implications of higher-order beliefs in monetary models (e.g. Woodford (2002), Amato and Shin (2006)). This paper abstracts from the complications associated with higher order beliefs by appealing to bounded rationality. Higher order beliefs may be important in some settings and we leave further study of this issue to future research.

2.2 Households

Agents have instantaneous utility functions given by $u\left(C_t^i, \frac{M_t^i}{P_t}\right) - v\left(N_t^i\right)$, where U is separable and C^i, P are the CES aggregators

$$C^i = \left(\int C^i(j)^{\frac{\theta-1}{\theta}} dj\right)^{\frac{\theta}{\theta-1}} \text{ and } P = \left(\int P(j)^{1-\theta} dj\right)^{\frac{1}{1-\theta}}. \tag{1}$$

Agents choose sequences for consumption, money holdings, labor, and goods prices to maximize expected discounted utility subject to a flow constraint. However, as is standard in the bounded rationality and learning literature, agents do not necessarily observe current values of endogenous variables when making decisions: it may be that $E_t^\tau x_t \neq x_t$. Quite naturally, we do assume that current values of an individual agent's own choice variables are known. Putting these assumptions together we find that when making optimal decisions, agent i, using expectations operator E^τ, faces a *perceived nominal flow constraint* as given by

$$C_t^i E_t^\tau P_t + M_t^i + B_t^i + I_{pt} = P_t^i Y_t^i + (1 + i_{t-1}) B_{t-1}^i + M_{t-1}^i + I_{rt}, \tag{2}$$

where P^i is the price set by agent i, Y^i is the quantity of goods produced by agent i, M^i is the money held by agent i, B^i are the bonds held by agent i, and I_p and I_r are nominal payments to and receipts from an insurance agency respectively, which the agent takes as given. The behavior of the insurance agency will be discussed in more detail below. Finally, the agent faces the additional constraints that the pair (Y_t^i, P_t^i) be chosen to lie on the relevant demand curve, and that changing prices P_t^i is possible with probability $1 - \gamma$, determined according to the Calvo lottery as is standard in New Keynesian models.

Under rationality, the theory of dynamic programing implies that agents will choose consumption bundles to satisfy an intertemporal Euler equation of the form

$$u_c(C_t^i) = \beta(1 + i_t) E_t^i \left(\frac{P_t}{P_{t+1}}\right) u_c\left(C_{t+1}^i\right), \tag{3}$$

as well as an associated transversality condition,

$$\lim_{k\to\infty} E_t^i \beta^{t+k} u_C(C_{t+k}^i) \frac{B_{t+k}^i}{P_{t+k}} \leq 0. \tag{4}$$

For general subjective expectations E_t^i it is not obvious that these conditions are necessary or sufficient for a consumption sequence to be optimal: the theory of dynamic programming and the principle of optimality were developed in stochastic environments assuming E_t^i are conditional expectations, and in the present setting, it is not clear that the relevant theorems apply. Instead, we take as a primitive behavioral assumption that those agents with subjective (non-rational) expectations choose a household plan that satisfies the Euler equations, and we require that along any equilibrium path, the associated transversality condition is satisfied.

The justification for these assumptions is straightforward: agents behave in a way to equate their expected marginal rate of substitution with the expected marginal rate of transformation, and so that they do not plan to accumulate valuable assets faster than they are discounted. We note that these assumptions, while usually implicit, are standard in the learning literature: see for example (Bullard and Mitra, 2002) and (Honkapohja and Evans, 2003). Indeed, it is hard to imagine settings where it would be reasonable to model agents as violating either type of condition. Additionally, we assume for simplicity that i_t is known to agents at time t by having the central bank announce its policy. Note that by separability of U, U_c is independent of real money balances.

Preston (2006) presents an interesting alternative approach, in a representative agent environment, where the agents' optimal plans also respect their perceived lifetime *intertemporal* budget constraint. Preston demonstrates that if subjective expectations satisfy the law of iterated expectations, then it is possible to represent the optimal plan as the solution to an Euler equation for appropriately defined expectations. For simplicity, we follow much of the learning literature and adopt the approach outlined in the previous paragraph, in which the Euler equations are assumed to hold for all t, and the transversality condition (and hence satisfaction of the lifetime budget constraint) is imposed ex-post. Explicitly incorporating long-horizon expectations into the model is the next step and is a topic of current research.

We follow Woodford (2003) in assuming the cashless limit case: the utility function is such that each household holds an arbitrarily small amount of real-money so that the central bank can effectively choose a target level for the interest rate and, although the real-money demand function dictates the size of open market operations consistent with that target, the actual effect of these operations on the net supply of bonds is negligible. In the current environment, the money-demand equation is given by

$$\frac{u_M(C_t^i, M_t^i/P_t)}{u_C(C_t^i, M_t^i/P_t)} = \frac{i_t}{1+i_t}. \tag{5}$$

Notice as well that the expression for money demand (5) does not depend explicitly on agents' expectations.[4]

We adopt the Calvo pricing structure, where with a positive probability each farmer's price may remain fixed. Acting as price setters, individual agents face the risk associated with this Calvo lottery.[5] The standard treatment allows agents to hedge

[4]In incomplete market settings i_t may depend on the distribution of agents' wealth holdings (e.g. Krusell and Smith (1998), Aiyagari (1994, 1995)). Our modeling of risk-sharing below makes wealth holdings a fraction of the real-income of their type cohort.

[5]For the purposes of this paper, we abstract from modeling aggregate risk.

against this risk by assuming the presence of complete markets; in this way, agents are able to guarantee themselves the average real income each period. Our model posits two types of agents. While each type will face an equal probability of being allowed to change prices, their expected real income differs due to heterogeneity of expectations. Because of this, deriving from micro-foundations a reduced form similar to the rational expectations case requires some care. For example, the derivation of the IS relation, which is obtained by log-linearizing the agents' consumption Euler equation, requires agents of a given type form expectations of their own consumption index, and the value of this index may differ from the index value of other types of agents; it is not immediate that aggregation across types, which is needed to obtain the model's reduced form equations, is possible in this case.

To address this issue, we assume the presence of a benevolent financial regulator who takes insurance premiums optimally from each type of consumer and then returns the proceeds in order to provide agents the means to hedge the risk associated to the Calvo lottery. The planner collects all income and then redistributes to each type of agent the average income of that agent's type. The planner does this so that each agent type, as indexed by their expectation formation mechanism, is fully insured against the risk associated to the possibility that they will not be tapped to adjust prices; in this way agents are able to guarantee themselves within type average real income.[6]

As an alternative there is a long history of studying heterogeneity in dynamic models with incomplete financial markets. In many cases, the existence of heterogeneous consumers and incomplete markets imply that equilibrium prices and allocations differ from the representative agent model, but the quantitative differences seem small (Krusell and Smith (1998)). In our setting, heterogeneity arises because agents use different expectations operators when solving for their optimal plans, thus their optimal allocations could be different. This highly stylized setting is designed to remain close, in reduced form, to the standard model. We thus view our model as a lower bound on the implications of heterogeneity.

Preston (2005, 2006) develops a micro-founded model with arbitrary subjective expectations and incomplete financial markets. In his formulation, agents engage in risk-sharing through labor markets and firm profit sharing. Importantly, in Preston's approach agents must respect their intertemporal budget constraint implying that long-horizon expectations matter: solutions to the Euler equation do not necessarily characterize optimal plans with subjective expectations. In our model, markets are also incomplete and, as an alternative to the use of an insurance planner, we could follow Preston, in which case long horizon expectations would matter. We expect distinct results with the long-horizon approach developed by Preston, and further,

[6]Our approach is similar to Kocherlakota (1996) except we assume there are no commitment issues, on either side, and this essentially creates two types of representative agents. This risk-sharing mechanism is also employed by Shi (1999) and Mankiw and Reis (2006).

we anticipate that with a fraction of rational and a fraction of long-horizon agents the instability results below will be maintained or strengthened.

Under our financial structure, agents of type τ may guarantee themselves a real income Ω^τ as given by

$$\Omega^1 = \frac{1}{\alpha P}\int_0^\alpha P(i)Y(i)di \quad \text{and} \quad \Omega^2 = \frac{1}{(1-\alpha)P}\int_\alpha^1 P(i)Y(i)di. \tag{6}$$

Notice that

$$P(\alpha\Omega^1 + (1-\alpha)\Omega^2) = \int P(i)Y(i)di = PY,$$

from which it follows that $Y = \alpha\Omega^1 + (1-\alpha)\Omega^2$.

In equilibrium, if agent i is of type τ then, $I^i_{rt} = P_t\Omega^\tau_t$ and $I^i_{pt} = P^i_t Y^i_t$. Therefore, in equilibrium,

$$C^i_t + b^i_t = \left(\frac{1+i_{t-1}}{1+\pi_t}\right) b^i_{t-1} + \Omega^\tau_t, \tag{7}$$

where b^i is the quantity of real bonds held by agent i, and $1+\pi_t = P_t/P_{t-1}$. A first order expansion of (7) around steady state, incorporating A1, yields

$$C^i_t - \bar{C}^i + b^i_t = \left(\frac{1+\bar{i}}{1+\bar{\pi}}\right) b^i_{t-1} + \Omega^\tau_t - \bar{\Omega}^\tau,$$

where we have exploited $\bar{b}^i = 0$. Using $\beta(1+\bar{i}) = 1$ and $\bar{C}^i = \bar{\Omega}^\tau = \bar{Y}$, we obtain the log-linear approximation

$$c^i_t = \hat{\Omega}^\tau_t \equiv \omega^\tau_t + \beta^{-1}\frac{b^i_{t-1}}{\bar{Y}} - \frac{b^i_t}{\bar{Y}}, \tag{8}$$

where c and ω are in log deviations from steady-state form, $c = log(C^i/\bar{Y}), \omega = log(\Omega^\tau/\bar{\Omega}^\tau)$.

To obtain a New Keynesian IS relation, we log-linearize (3):

$$c^i_t = E^i_t c^i_{t+1} - \sigma^{-1}\left(i_t - E^i_t\pi_{t+1}\right). \tag{9}$$

We note that $E^i_t\pi_{t+1} = E^i_t(\log P_{t+1} - \log P_t) = E^i_t \log P_{t+1} - E^i_t \log P_t$, and may not be equal to $E^i_t(\log P_{t+1}) - \log P_t$, in the event P_t is not observable.

Inserting (8) into (9) yields the equation

$$\hat{\Omega}^\tau_t = E^\tau_t\hat{\Omega}^\tau_{t+1} - \sigma^{-1}(i_t - E^\tau_t\pi_{t+1}). \tag{10}$$

Note that along any equilibrium path, equation (10) must be satisfied for both agent types. The IS curve comes from aggregating this equation across all households. Iterate this equation forward, while employing the assumptions on agents' expectations, to obtain

$$\hat{\Omega}^\tau_t = \hat{\Omega}^\tau_\infty - \sigma^{-1}E^\tau_t\sum_{k\geq 0}\left(i_{t+k} - \pi_{t+k+1}\right), \tag{11}$$

where $\hat{\Omega}^\tau_\infty = \lim_{k\to\infty} E^\tau_t \hat{\Omega}^\tau_{t+k}$.[7] Noting that bond market clearing requires $\alpha b^1_t = -(1-\alpha)b^2_t$, we get that

$$\alpha\hat{\Omega}^1_t + (1-\alpha)\hat{\Omega}^2_t = y_t.$$

Then

$$\begin{aligned} y_t &= \alpha\hat{\Omega}^1_t + (1-\alpha)\hat{\Omega}^2_t \\ &= \alpha\hat{\Omega}^1_\infty + (1-\alpha)\hat{\Omega}^2_\infty - \sigma^{-1}\hat{E}_t \sum_{k\geq 0}(i_{t+k} - \pi_{t+k+1}) \\ &= \hat{E}_t y_{t+1} - \sigma^{-1}\left(i_t - \hat{E}_t\pi_{t+1}\right) \\ &\quad + \alpha\hat{\Omega}^1_\infty + (1-\alpha)\hat{\Omega}^2_\infty - \hat{E}_t\left(\alpha\hat{\Omega}^1_\infty + (1-\alpha)\hat{\Omega}^2_\infty\right), \end{aligned}$$

where $\hat{E} = \alpha E_1 + (1-\alpha)E_2$. Here we note that the final equality makes use of the law of iterated expectations at the aggregate level: see assumption (A5).

To obtain an IS form most similar to the homogeneous expectations case, we make the additional assumption that all agents predict limiting wealth in an identical manner:

A6. All agents have common expectations on expected differences in limiting wealth:

$$\alpha\hat{\Omega}^1_\infty + (1-\alpha)\hat{\Omega}^2_\infty - \hat{E}_t\left(\alpha\hat{\Omega}^1_\infty + (1-\alpha)\hat{\Omega}^2_\infty\right) = 0.$$

Assumption (A6) can be written equivalently as $\hat{\Omega}^j_\infty = E^{j'}\hat{\Omega}^j_\infty, j' \neq j$. While this is plausible, and is satisfied when agents have rational expectations regarding limiting wealth, one might envision agents holding beliefs that violate it. In that case, the distribution of wealth dynamics will affect equilibrium allocations. Such an analysis is beyond the scope of the present paper but is an interesting topic for further research.

The above analysis provides the following proposition:

Proposition 1 *If agents' expectations E^1 and E^2 satisfy assumptions (A1)-(A6) then, up to a log-linear approximation, equilibrium output and inflation satisfy the following IS relation*

$$y_t = \hat{E}_t y_{t+1} - \sigma^{-1}\left(i_t - \hat{E}_t\pi_{t+1}\right) \tag{12}$$

where $\hat{E} = \alpha E^1 + (1-\alpha)E^2$.

From this we conclude that with heterogeneous expectations the IS equation is the usual IS curve where conditional expectations are replaced with a convex combination of the heterogeneous expectations operators.

[7] We take $\hat{\Omega}^\tau_\infty$, which is the time t expected limiting value of log consumption, to be finite and independent of time.

2.3 Optimal Pricing

To determine the equilibrium behavior of inflation we turn to the agents' pricing problem. We first compute demand for a particular good: the CES aggregate consumption index implies that demand by an agent of type τ for good i, is given by $(P_i/P)^{-\theta}\left(\Omega^\tau + (1+i_{-1})B^\tau_{-1}/P - B^\tau/P\right)$. Aggregating across type and imposing bond market clearing yields

$$Y^i = \alpha\,(P_i/P)^{-\theta}\,\Omega^1 + (1-\alpha)\,(P_i/P)^{-\theta}\,\Omega^2 = (P_i/P)^{-\theta}\,Y$$

as usual.

Note that because of the presence of the insurance company, an agent's income is, in effect, independent of his effort; because of this, an agent's optimal output Y^i is zero. To avoid this free-rider problem, we assume that private agents are contracted to choose price and output as if they faced their perceived constraint (2), and we further assume that these contracts are fully enforceable. In some sense, this requires the agent to behave as if they will receive their full marginal revenue from producing more.[8] With this assumption, we may proceed as follows: Let agent i be of type τ. Use the perceived budget constraint to write $C^i_{t+k} = C^i_{t+k}(P^i_t)$, in the event that agent i can not change prices for k periods. Then P^i_t is chosen by contract to solve

$$\max E^\tau_t \sum_{k\geq 0} (\beta\gamma)^k \left[u(C^i_{t+k}(P^i_t),\cdot) - v\left(\left(P^i_t/P_{t+k}\right)^{-\theta} Y_{t+k}\right)\right]. \tag{13}$$

As above, because expectations are not necessarily rational, we assume rather than infer that agents behave in a manner consistent with the usual first order conditions of this pricing problem. This FOC may be log-linearized as usual, but now there is the presence of both Y and C^i. Passing expectations across choice variables (so that $\hat{\Omega}^\tau_t$ and not $E^\tau_t\hat{\Omega}^\tau_t$ enters the equation) and imposing equilibrium yields

$$E^\tau_t \sum_{k\geq 0}(\gamma\beta)^k \left(\log\left(P^i_t\right) - \log\left(P_{t+k}\right) - \zeta_1\hat{\Omega}^\tau_{t+k} - \zeta_2 y_{t+k}\right) = 0.$$

Subtract $E^\tau_t \log P_t$ from each side and solve for $\log P^i_t - E^\tau_t \log P_t$. We obtain[9]

$$\log P^i_t - E^\tau_t \log P_t = E^\tau_t \sum_{k\geq 0}(\gamma\beta)^k \left(\gamma\beta\pi_{t+k+1} + (1-\gamma\beta)(\zeta_1\hat{\Omega}^\tau_{t+k} + \zeta_2 y_{t+k})\right).$$

[8] One could equivalently imagine a household where one member consumes given the insurance payment and the other works and makes optimal pricing decisions.

[9] This step requires the following observation: Let $s_t = \sum_{k=1}^t \pi_k = \log P_t - \log P_0$, and set $s_0 = 0$. Then

$$(1-\delta)\sum_{t\geq 0}\delta^t(\log P_t - \log P_0) = \sum_{t\geq 0}\delta^t s_t - \sum_{t\geq 0}\delta^{t+1}s_t = \sum_{t\geq 0}\delta^t\pi_{t+1}.$$

Stepping this equation forward, applying the L.I.E., using the linearity assumption (A2) and rearranging yields

$$\log P_t^i - E_t^\tau \log P_t = \gamma\beta E_t^\tau \pi_{t+1} + (1-\gamma\beta)\left(\zeta_1 \hat{\Omega}_t^\tau + \zeta_2 E_t^\tau y_t\right) + \gamma\beta E_t^\tau \log P_{t+1}^i / P_{t+1}. \quad (14)$$

We turn now to the evolution of aggregate prices. The pricing decision is homogeneous within type, so we now say P_t^τ is the optimal price chosen by an agent of type τ in time t, and $P_t^\tau(j)$ denotes the price set by firm j (of type τ) in time t. Then

$$(P_t)^{1-\theta} = \int_0^\alpha P_t^1(j)^{1-\theta} dj + \int_\alpha^1 P_t^2(j)^{1-\theta} dj.$$

The proportion $1-\gamma$ of each type changes their optimal price in period t. Thus

$$\int_0^\alpha P_t^1(j)^{1-\theta} dj = (1-\gamma)\alpha \left(P_t^1\right)^{1-\theta} + \gamma \int_0^\alpha P_{t-1}^1(j)^{1-\theta} dj,$$

and similarly for $\int_\alpha^1 P_t^2(j)^{1-\theta} dj$. We obtain

$$(P_t)^{1-\theta} = (1-\gamma)\alpha \left(P_t^1\right)^{1-\theta} + (1-\gamma)(1-\alpha)\left(P_t^2\right)^{1-\theta} + \gamma \left(P_{t-1}\right)^{1-\theta}.$$

Log linearization yields

$$p_t = (1-\gamma)\alpha p_t^1 + (1-\gamma)(1-\alpha)p_t^2 + \gamma p_{t-1},$$

so that, subtracting $(1-\gamma)p_t$ from both sides, we have

$$\alpha \log P_t^1/P_t + (1-\alpha)\log P_t^2/P_t = \frac{\gamma}{1-\gamma}\pi_t.$$

Finally, multiply equation (14) by α for $\tau = 1$ and by $1-\alpha$ for $\tau = 2$ and add to get the following result:

Proposition 2 *If agents' expectations E^1 and E^2 satisfy assumptions (A1)-(A6) then, up to a log-linear approximation, equilibrium output and inflation satisfies the AS relation*

$$\pi_t = \beta \hat{E}_t \pi_{t+1} + \lambda_1 y_t + \lambda_2 \hat{E}_t y_t + \frac{1-\gamma}{\gamma}(\hat{E}_t \pi_t - \pi_t). \quad (15)$$

where $\hat{E} = \alpha E^1 + (1-\alpha)E^2$.

Here $\lambda = \lambda_1 + \lambda_2$ is the usual coefficient on output gap in the new Keynesian Phillips curve. Notice that if all agents are rational, so that $\hat{E}_t = E_t$, this curve reduces to the usual AS equation.

It is worth remarking on an important difference between (15) and the usual New Keynesian expectational Phillips curve. Equation (15) includes two terms that incorporate current expectations of the current state variables, $\hat{E}_t y_t, \hat{E}_t \pi_t$. These terms arise because of the natural timing assumptions for boundedly rational agents. We assumed, as is standard in the adaptive learning literature, that boundedly rational agents observe current exogenous variables but contemporaneous endogenous state variables are unobserved. In price-setting this implies that agents do not see the aggregate price when they set their own price, and similarly for observing aggregate output. This assumption is often employed for adaptive agents to avoid simultaneity in beliefs and outcomes that, while natural in a rational expectations equilibrium, seems less plausible for boundedly rational agents. If these state variables were observed then the final term in (15) would reduce to zero and $\lambda_1 y_t + \lambda_2 \hat{E}_t y_t = \lambda y_t$, and we would have a heterogeneous expectations Phillips curve with the same reduced form as under RE. We anticipate that the form of (15) may have implications for an estimated version of the model and any potential empirical implications are the topic of future research.

2.4 Heterogeneous Expectations Equilibria

The aggregate dynamics of our model are given by

$$\begin{aligned} y_t &= \hat{E}_t y_{t+1} - \sigma^{-1} \left(i_t - \hat{E}_t \pi_{t+1} \right) \\ \pi_t &= \beta \hat{E}_t \pi_{t+1} + \lambda_1 y_t + \lambda_2 \hat{E}_t y_t + \frac{1-\gamma}{\gamma} (\hat{E}_t \pi_t - \pi_t), \end{aligned}$$

and it is tempting to think that, given a policy process i_t, an output gap and inflation process y_t, π_t satisfying the above system constitutes an equilibrium to our model; however, we must proceed with care to ensure that all of the assumptions regarding optimal agent behavior are respected. The definition of equilibrium must include the restriction that $\hat{\Omega}_t^\tau$ satisfies the Euler equation (10) for each agent type τ, and that the transversality condition of each agent type is respected. At the current level of generality, little more can be said about the nature of the model's equilibrium; but we will return to these issues in Section 3 below where, for our application to monetary policy and determinacy, we explicitly define the expectations operators.

2.5 Further Discussion

The above analysis constructed a New Keynesian model with heterogeneous expectations. We demonstrated that, under reasonable assumptions on expectations, individual decision rules aggregate into heterogeneous expectations IS and AS relations.

Interestingly, the heterogeneous expectations Phillips curve incorporates current expectations of contemporaneous output and inflation. These terms arise if some agents form decisions based on forecasts of contemporaneous endogenous state variables. These aggregation results are new and are the primary contribution of this paper. This subsection discusses at greater length some of the assumptions required for aggregation.

The aggregation results, captured in Propositions 1 and 2, rely heavily on the law of iterated expectations, both at the individual level (assumption (A4)) and at the aggregate level (assumption (A5)), and some discussion of the application of this law is warranted. For clarity, we impose the law of iterated expectations as an axiom. It is, however, a consequence of consistent behavior on the part of agents: we imagine expectations of a given agent as corresponding to subjective probabilities of future events, and, because of this correspondence, the law of iterated expectations holds by definition. Indeed, it is always possible to invent the required subjective probability distribution that rationalizes a forecasting rule. Even if agents maximize utility subject to an incorrect subjective probability distribution, their beliefs still satisfy the L.I.E. with respect to this distribution.

One can construct beliefs that satisfy the law of iterated expectations by building forecasting models recursively by forward iteration, so that the law of iterated expectations holds by construction. This is the standard assumption in the literature on learning in which expectations of future variables are based on forward iteration of the agents' perceived laws of motion: see for example, (Bullard and Mitra, 2002) and (Evans and Honkapohja, 2003). Also, it is precisely this construction that we adopt in the next section when imposing adaptive expectations on a proportion of agents.

While we contend that the law of iterated expectations is a defining property of expectation formation, other assumptions were made primarily for simplicity. For example, our risk-sharing arrangements were made to identify consumption with expected within-type wealth net bond holdings. It would be interesting for future research to study the insurance markets and incomplete financial markets in greater detail and to examine the wealth distribution effects on aggregate variables arising from heterogeneous expectations. Along these same lines, assumption (A6), regarding expected limiting wealth, also simplifies the analysis and eases aggregation. We think that if we were to model the wealth distribution directly there would be even starker differences from the standard rational expectations model. For these reasons, one interpretation of our results is that a model with heterogeneous expectations could lead to important differences from a representative agent model.

3 An Application to Monetary Policy

Propositions 1 and 2 constitute a New-Keynesian laboratory in which the impact of expectations heterogeneity on equilibrium outcomes can be studied and compared to their rational expectations counterparts. In this section we provide one example of our model's potential for providing new and interesting results by examining its determinacy properties. We anticipate that there are other applications of our framework. Particularly, as alluded to in the introduction, when optimal policy is viewed as a management of expectations, one might expect that the precise form of an optimal policy rule will hinge on the distribution of heterogeneity. Such an analysis is beyond the scope of the present paper. Instead we provide an example that demonstrates the potential for heterogeneity to alter the dynamic properties of the economy.

Our focus will be on equilibrium determinacy. A linear rational expectations model is said to be determinate if there is a unique non-explosive equilibrium, indeterminate if there are many non-explosive equilibria, and explosive otherwise, and we simply extend these definitions to the model with heterogeneous expectations.[10] That New Keynesian models, when closed with Taylor-type instrument rules, may exhibit indeterminacy has been noted by a number of authors: see, for example, Bernanke and Woodford (1997), Woodford (1999) and Svensson and Woodford (2003). Indeterminacy is thought to be undesirable because of the presence of multiple equilibria, many of which may be welfare reducing. Clarida, Gali and Gertler (2000) have suggested that the volatile nature of the US time-series in the 1970s can, in part, be explained by the presence of sunspot equilibria. Further empirical evidence is provided by Lubik and Schorfheide (2005).

In a univariate reduced form model having the same structure as the heterogeneous NK model – see (21) below – Branch and McGough (2005) found that the presence of adaptive agents impacts the determinacy properties of the model. For some parameterizations, the presence of adaptive agents may be stabilizing in the sense that a model that is indeterminate under the assumption of uniform rationality may be determinate when some agents form their expectations adaptively. The converse also holds: for some parameterizations even if only a very small proportion of agents are adaptive, an otherwise determinate model may be indeterminate.

The importance from a policy perspective of understanding a monetary model's determinacy properties, together with the possibility demonstrated by Branch and McGough (2005) that the presence of adaptive agents can impact determinacy, motivates our analysis of the determinacy properties of the New Keynesian model with heterogeneous expectations. The presence of heterogeneous agents raises an obvious question: how are the determinacy properties of a such a model computed? This

[10] Non-explosive equilibria are the focus of rational expectations model because explosive time series typically violate transversality.

question was answered in Branch and McGough (2005): specifically, given a monetary policy $\{i_t\}$, and a pair of expectations operators (E^1, E^2), solutions to (12), (15) are what Branch and McGough (2005) define as *Heterogeneous Expectations Equilibria.* In this earlier paper, we showed that any model with heterogeneous expectations of a particular form can be re-written in terms of an associated rational expectations model, bringing to bear the standard rational expectations toolkit to analyze the number and nature of heterogeneous expectations equilibria. In this way, the analysis of determinacy in our model reduces to the well-understood determinacy analysis of the associated rational expectations model.

Before analyzing determinacy, we close the heterogeneous expectations New Keynesian model by specifying a monetary policy rule and a precise form for private agents' expectations. We turn to these tasks in the following two subsections.

3.1 Monetary Policy

The central bank is assumed to use a forward-looking Taylor rule to set interest rates:

$$\mathrm{PR}_1 : i_t = \alpha_y E_t y_{t+1} + \alpha_\pi E_t \pi_{t+1}, \tag{16}$$

where expectations here are taken to be rational. This expectations based policy rule has been studied by Bullard and Mitra (2002), Evans and Honkapohja (2003a,b), Evans and McGough (2005a,b), and Preston (2005). It has the advantage of conditioning on observables, and thus addresses the concern of McCallum that contemporaneous policy rules are not realistic in practice; and it imposes that policy-makers set their instrument conditional on their best forecasts of future inflation and output gap (e.g. Svensson and Woodford (2003)): in this way, policy makers can account for the lag between a policy's implementation and its impact.

We adopt (16) as our benchmark policy rule; however, we also consider the following specification of the policy rule:

$$\mathrm{PR}_2 : i_t = \alpha_y \hat{E}_t y_{t+1} + \alpha_\pi \hat{E}_t \pi_{t+1}. \tag{17}$$

Here, policy makers react to private agents expectations rather than to optimal forecasts; this is consistent with the view that policy-makers set policy to anchor agents' expectations. In particular, Bernanke (2004) advocates policy that reacts aggressively to private-sector expectations, particularly when they are non-rational. Evans and Honkapohja (2003) show that such a rule may have good properties in the presence of boundedly rational agents.

3.2 Rational versus Adaptive Expectations

In order to present precise results concerning our model's determinacy properties, we must make assumptions on $\hat{E}_t$. For simplicity, we focus on rational versus simple adaptive expectations. More specifically, we make the following assumption on beliefs: agents of type 1 are rational and agents of type 2 are adaptive on output and inflation: thus for $x = y$ or π,

$$E_t^2(x_t) = \theta_x x_{t-1},$$

and, as is standard in the learning literature, $E_t^2 x_{t+k}$ is constructed iteratively. Having defined adaptive expectations on x_{t+k}, we impose that the operator satisfy the assumptions (A1)-(A6). Note that from our construction, adaptive expectations satisfies the L.I.E. However, somewhat more subtly, rational agents are not "hyper-rational" in the following sense: hyper-rationality would imply that $E_t^1 E_{t+1}^2 x_{t+1} = \theta_x x_t$; however, assumption (A5) requires that $E_t^1 E_{t+1}^2 x_{t+1} = E_t^1 x_{t+1}$.[11]

Using (A5) we obtain

$$\begin{aligned}\hat{E}_t x_t &= n_x x_t + (1 - n_x)\theta_x x_{t-1} \\ \hat{E}_t(x_{t+1}) &= n_x E_t x_{t+1} + (1 - n_x)\theta_x^2 x_{t-1}.\end{aligned}$$

Here n_x is the proportion of agents having rational expectations. The operator E_t^2 is a form of adaptive ($\theta < 1$) or extrapolative ($\theta > 1$) expectations (for simplicity, below we refer to E^2 as an "adaptive" expectations operator, even in the event $\theta > 1$). Such expectations can be thought of as arising from a simple linear perceived law of motion $x_t = \theta x_{t-1}$. Under certain conditions, including the assumption of homogeneous expectations, it is possible for real-time estimates of θ to converge to their REE value. In this paper, we take the value of θ as given and leave to future research the study of stability under learning of heterogeneous expectations equilibria.[12] We do, however, let the fraction of rational agents and the adaptive parameter θ differ by forecasting variable.

Adaptive expectations of this form have been considered by Brock and Hommes (1997, 1998), Branch (2002), Branch and McGough (2005), and Pesaran (1987). When $\theta = 1$ then the operator is usually called 'naive' expectations, $\theta < 1$ are adaptive in the sense that they dampen recent observations, $\theta > 1$ are often called 'extrapolative' or trend chasing expectations. The $\theta > 1$ is given particular emphasis by Brock and Hommes (1998).

Despite the simple form of our heterogeneous expectations operator, there is evidence in survey data on inflation expectations that agents are distributed across rational expectations and adaptive expectations of this form (see Branch (2004)). The contribution of this paper, however, is to show how to incorporate heterogeneous expectations into a monetary model and show that their presence can have significant implications for monetary policy. Future research should undertake serious empirical

[11] More carefully, our rational agents are assumed to know the conditional distributions of output and inflation, but not the expectations of non-rational agents. We further assume that rational agents are not sophisticated enough to back out the expectations of non-rational agents.

[12] Some progress on this issue has been made by Guse (2005) in a simple univariate framework.

studies of heterogeneous beliefs in monetary DSGE models along the lines of Smets and Wouters (2003), Lubik and Schorfheide (2005), or Milani (2006).

3.3 Determining Determinacy

Our fully specified aggregate model is given by

$$\begin{aligned}
y_t &= \hat{E}_t y_{t+1} - \sigma^{-1}\left(i_t - \hat{E}_t \pi_{t+1}\right) && (18)\\
\pi_t &= \beta \hat{E}_t \pi_{t+1} + \lambda_1 y_t + \lambda_2 \hat{E}_t y_t + \frac{1-\gamma}{\gamma}(\hat{E}_t \pi_t - \pi_t) && (19)\\
i_t &= \alpha_y E_t y_{t+1} + \alpha_\pi E_t \pi_{t+1} && (20)\\
\hat{E}_t y_t &= n_y y_t + (1-n_y)\theta_y y_{t-1}\\
\hat{E}_t(y_{t+1}) &= n_y E_t^1 y_{t+1} + (1-n_y)\theta_y^2 y_{t-1}\\
\hat{E}_t \pi_t &= n_\pi \pi_t + (1-n_\pi)\theta_\pi \pi_{t-1}\\
\hat{E}_t(\pi_{t+1}) &= n_\pi E_t^1 \pi_{t+1} + (1-n_\pi)\theta_\pi^2 \pi_{t-1},
\end{aligned}$$

together with the alternative policy rule PR_2. Simplification yields the following reduced form model:

$$Fx_t = BE_t x_{t+1} + Cx_{t-1}, \tag{21}$$

for appropriate matrices F, B, and C, where $\det(B) \neq 0$ and $x = (y, \pi)'$. Equation (21) is the rational expectations model (the ARE) associated with the heterogeneous expectations model above. Since the ARE has the same form as a rational expectations model with predetermined variables there is an established toolkit for determining determinacy, thereby making our approach easily accessible to practitioners.[13] Techniques for analyzing the determinacy properties of a linear model are well-known: see for example Blanchard and Kahn (1980).

Let

$$M = \begin{pmatrix} B^{-1}F & -B^{-1}C \\ I_2 & 0 \end{pmatrix}. \tag{22}$$

The determinacy properties of the model depend on the magnitude of the eigenvalues of M, which we write as λ_i, where $i > j \Rightarrow |\lambda_i| \leq |\lambda_j|$.[14] Note that in general there are two predetermined variables, so that determinacy obtains when precisely two of the eigenvalues of M are outside the unit circle: fewer eigenvalues outside the unit circle implies indeterminacy and more implies explosiveness. As we will see in the sequel,

[13] As mentioned above, in rational expectations models, non-explosiveness is typically required in order to avoid violation of asymptotic first order conditions. Because a portion of our agents are rational and therefore have analogous asymptotic first order conditions, we also focus on non-explosive equilibria: see Sub-section 3.4 below.

[14] Additive noise in the IS or AS relation does not impact the determinacy properties.

the notion of indeterminacy can be refined in a useful way: following Evans and McGough (2005a), we say that if $|\lambda_1| < 1$ then the model is *order two indeterminate* (or, equivalently, *exhibits order two indeterminacy*) and if $|\lambda_1| > 1 > |\lambda_2|$ we say the model is *order one indeterminate*. Note that the order of the indeterminacy corresponds to the largest possible dimension of a martingale difference sequence that can coordinate an associated sunspot equilibrium: for a detailed discussion see Evans and McGough (2005a). Finally, the same analysis applies in case policy rule two is used with the only modification being the dependence of the matrix M on the model's deep parameters.

3.4 Individual agents' constraints

Analysis of the determinacy properties of the associated rational model, given by (21), is straightforward and one can proceed by employing the usual solution techniques. However, the heterogeneity in our model raises an additional concern: how do we know that the aggregate time-series determined by (21) can also arise by aggregating from household plans? That is, a heterogeneous expectations equilibrium must simultaneously solve (21) and the individual agents' first order conditions. Given a solution to (21), it is always possible to back out a household plan for each individual since aggregate output is a weighted average of consumption across types. However, it also must be verified that, along an equilibrium path, bond holdings satisfy the transversality condition for each agent.

The transversality condition requires that

$$\lim_k E_t^\tau \beta^k u_c(C_{t+k}^\tau) b_{t+k}^\tau \leq 0. \tag{23}$$

is satisfied, along an equilibrium path, at each t and for each agent type τ. We have not yet specified explicitly how type $\tau = 2$ "adaptive" agents forecast $u_c(C_{t+k}^2) b_{t+k}^2$. Since this is a forecast on the household side, and $\tau = 2$ are adaptive agents, it seems natural to assume they forecast this term via

$$E_t^2 u_c(C_t^2) b_t^2 = \hat{\theta} u_c(C_{t-1}^2) b_{t-1}^2,$$

for some value $\hat{\theta}$, and the forecast function is constructed iteratively as before. *A priori*, there is no reason to expect $\hat{\theta} = \theta_y = \theta_\pi$. Below, we restrict attention to the case where $\hat{\theta} = \theta_y$. It follows that (23) is satisfied for adaptive agents provided that $\hat{\theta} < 1/\beta$.

We also have to ensure rational agents satisfy their transversality condition. To check this condition, first forward iterate the Euler equation for rational agents, to obtain

$$\hat{\Omega}_t^1 = -\sigma E_t \sum (i_{t+1} - \pi_{t+k+1}). \tag{24}$$

Since i_t, π_t are aggregate processes that, along with y_t, are nonexplosive solutions to (21), it follows that we may write $i_{t+1} - \pi_{t+k+1} = Az_{t+k}$, where z_t is a nonexplosive VAR in current and lagged output and inflation.[15] It follows from (24) that $\hat{\Omega}_t^1$ is uniformly bounded provided the aggregate variables are uniformly bounded as well. Next, we note that Ω_t^1 is non-explosive since it is smaller than Y_t, which itself is non-explosive. Then $\omega_t^1 = \log(\Omega_t^1/\bar{\Omega}^1)$ is non-explosive. Since $\hat{\Omega}_t^1$ is non-explosive, it follows that $\beta^{-1}b_{t-1}^1 - b_t^1 = \hat{\Omega}_t^1 - \omega_t^1$ is non-explosive, so that b_t is expected to grow at most at a rate slower than β^{-1}. Thus, any process for bonds supporting a non-explosive equilibrium time series will satisfy the associated transversality conditions. We conclude, therefore, that given our specification for adaptive expectations, a nonexplosive solution to (21) is a heterogeneous expectations equilibrium.

3.5 Numerical Analysis of Determinacy

Numerical results require assigning values to the reduced form model's parameters, i.e. to $\lambda_1, \lambda_2, \gamma, \sigma$, and β; also the proportions of rational agents n_x and the values of the adaptation parameters θ_x must be specified. These parameters could be estimated, however, such an estimation is beyond the scope of this simple application. Instead, we "derive" values for our parameters using estimates available in the literature; and we test for robustness of our results to varying parameter values whenever possible.

The reduced form parameters of the usual New Keynesian model (i.e., the model under rational expectations) are given by λ (where $\lambda = \lambda_1 + \lambda_2$) and σ^{-1}. These parameters have been estimated by a number of authors including Woodford (1999), Clarida, Gali, and Gertler (2000) and McCallum and Nelson (1999):[16] see Table 1. We use these authors' estimate of σ^{-1}, and set, as our benchmark, $\lambda_i = \lambda/2$.[17] We test the robustness of this latter assumption by allowing the relative sizes of the λ_i to vary. There are also several estimates of γ available: we use as a benchmark $\gamma = .65$ as given by Walsh (2003), and again, test for robustness. The discount factor β is taken to be .99.

Given reduced form parameter values, the model is closed by specifying a policy rule (i.e. setting values for α_x and α_π), and choosing values for $\theta = (\theta_y, \theta_\pi)'$ and $n = (n_x, n_\pi)'$. To perform our analysis, we fix θ and then for $n_* \in \{1, .99, .9, .7\}$ we analyze the determinacy properties of models characterized by the following policy rules: $0 < \alpha_\pi < 2$, $0 < \alpha_x < 2$. We refer to this region as the benchmark policy space.

[15] The notion of "nonexplosiveness" used here can be made formal in a variety of ways depending on the type of extrinsic noise processes under consideration. For example, one can restrict attention to covariance stationary processes, or impose the weaker restriction of conditionally uniformly bounded processes. See Evans and McGough (2005c) for details.

[16] Calibrations are for quarterly data with i_t and π_t measured as quarterly rates. The CGG calibration (based on annualized rates) is adjusted accordingly

[17] It is straightforward to compute that $\lambda_i > 0$.

Table 1: Calibrations

Author(s)	σ^{-1}	λ
W	1/.157	.024
CGG	4	.075
MN	.164	.3

Finally, in the sequel, we will say that certain numerical results "obtain robustly." By this, we will mean that they obtain under the following conditions:

- For both policy rules;
- Across calibrations for benchmark γ and λ_i, and for specified θ and n;
- Across $\lambda_2 \in \{.005, .012, .022\}$, for the W-calibration, benchmark γ, and for specified θ and n;
- Across $\gamma \in \{.4, .65, .9\}$, for the W-calibration, benchmark λ_i, and for specified θ and n.

3.5.1 Adaptive Expectations: $\theta < 1$.

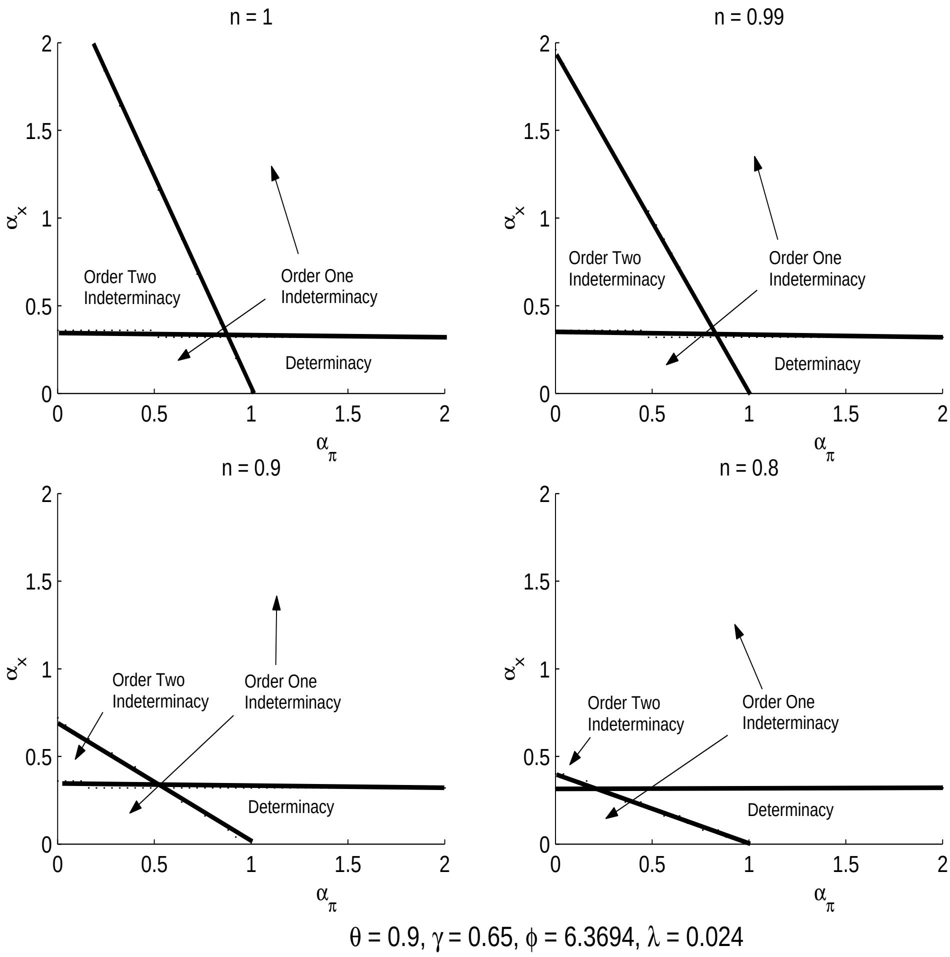
We first consider the case where adaptive agents place less weight on past data: that is, when $\theta < 1$. Figure 1 presents results for the Woodford calibration under the benchmark assumptions, with $\theta = .9$.[18] The NW panel indicates the outcome under rationality, and is consistent with the results in Evans and McGough (2005). Much of the parameter space corresponds to indeterminacy of both order one and order two, with determinacy prevailing only for large α_π and low α_x. Now notice that as n decreases, the region corresponding to determinacy increases in size (as a proportion of the benchmark policy space). Thus we obtain the following result.

Result 3 *If $\theta < 1$ then policy rules corresponding to indeterminacy when $n = 1$ may yield determinacy when there is even a small proportion of adaptive agents in the economy. In this sense, the presence of adaptive agents may be stabilizing.*

As indicated in the NW panel, with full rationality, indeterminacy comprises a large portion of the parameter space. To avoid indeterminacy, the Taylor principle instructs policy-makers to "lean against the wind," by setting the coefficient on inflation (or expected inflation) in the interest rate rule larger than one. In this way, a

[18]For all of our examples, $n_y = n_\pi$.

Figure 1: Determinacy Properties: $\theta < 1$.



rise in inflation is met with a larger rise in the nominal rate, and through the Fisher relation this implies a rise in the real interest rate that will have a dampening effect on the economy.

While the Taylor principle does, in many cases, render a model determinate, and while it is wonderfully elegant and wholly intuitive, it is known in general to be neither necessary nor sufficient to guarantee a unique equilibrium: Bullard and Mitra (2002) contradict both necessity and sufficiency in a purely forward looking New Keynesian model; other examples include Honkapohja and Mitra (2005) and Evans and McGough (2005a, 2005b).[19]

[19]Benhabib, Schmidt-Grohe and Uribe (2003) show that even when it does yield determinacy, a rule satisfying the Taylor principle may by ill-advised.

Both the Taylor principle, and the fact that it may be neither necessary nor sufficient for determinacy is evident in Figure 1. The NW panel contains a steep, negatively sloped line anchored to the horizontal axis at $\alpha_\pi = 1$. For $\alpha_x = 0$, this line marks the boundary between rules that satisfy the Taylor principle and those that do not. This panel shows that rules which satisfy the Taylor principle, but which also set $\alpha_x > \approx .35$ result in order one indeterminacy: here the Taylor principle is not sufficient to prevent indeterminacy. It is also evident from the NW panel, that for values of α_π which are smaller than but very near to one, there correspond α_x so that the model is determinate, thus demonstrating that the Taylor principle is not necessary for determinacy to obtain.

Now notice the impact of heterogeneity: as n decreases the line anchored at $\alpha_\pi = 0$ rotates counterclockwise, thereby enlarging the relative size of the determinacy region; this rotation therefore increases the number of rules that simultaneously violate the Taylor principle and yield determinacy. We find that this *counterclockwise* rotation of the line anchored at $\alpha_\pi = 1$ obtains robustly.

3.5.2 Extrapolative Expectations: $\theta > 1$.

We now turn to the case where agents place greater weight on past data, i.e. $\theta > 1$. Here there is an additional subtlety: for non-explosive time paths of output gap and inflation to correspond to optimal behavior on the part of adaptive agents, it must be the case that $\hat{\theta}$, the parameter used by adaptive agents to forecast the current value of future bond holdings, be less that β^{-1}. To construct Figure 2, we take $\theta_\pi = \theta_y = \hat{\theta} = \beta^{-1} - \varepsilon$, for some $\varepsilon > 0$, and we consider a case in which $\theta_\pi \neq \theta_y = \hat{\theta}$ in Figure 3.[20] Figure 2 presents results for the benchmark parameter values. The NW panel is identical to the panel in Figure 1, but here, as n decreases, the line anchored at $\alpha_\pi = 1$ rotates *clockwise*, thereby reducing the relative size of the region in parameter space corresponding to determinacy. However, for sufficiently high fractions of adaptive agents (small n) the presence of heterogeneity may be stabilizing, in accordance with Result 3. This leads us to the next result.

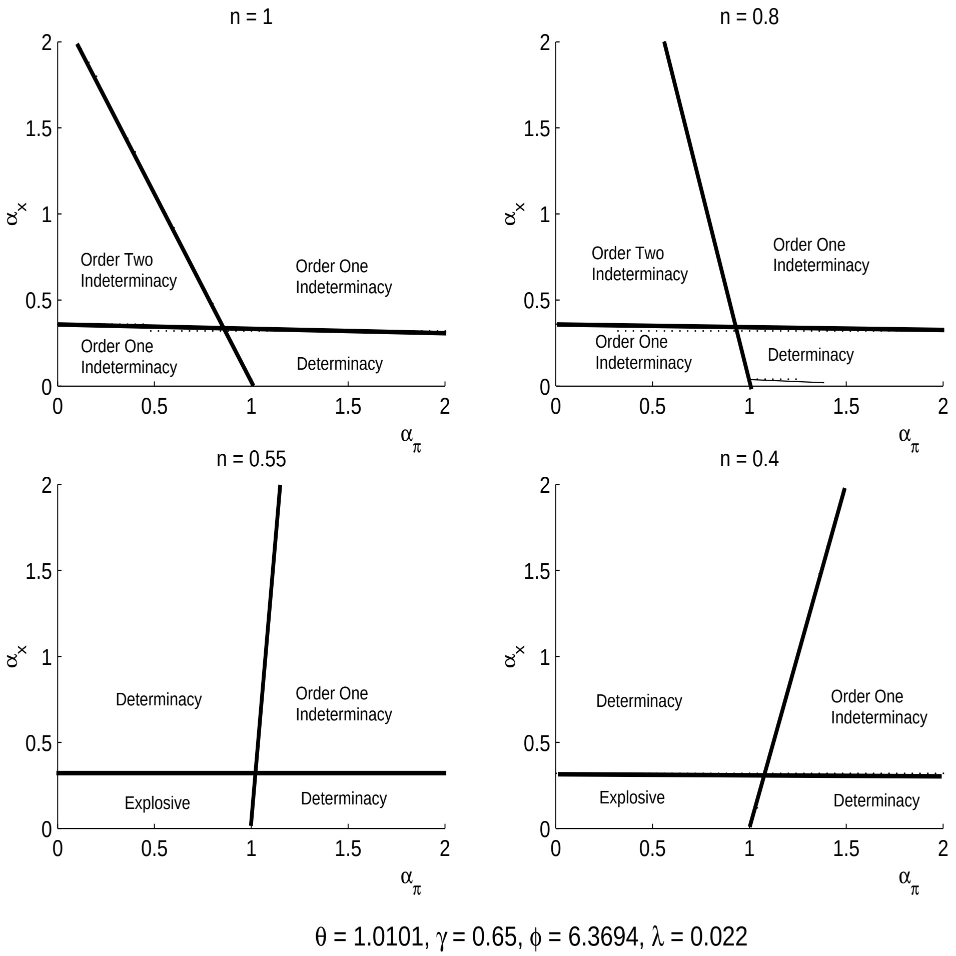
Result 4 *If $\theta > 1$ then policy rules that yield a determinate outcome in case of rationality may present indeterminacy in case even a very small proportion of agents are behaving adaptively. In this case, we find that the presence of adaptive agents may be destabilizing.*

We find that this clockwise rotation line anchored at $\alpha_\pi = 1$ qualitatively obtains robustly.[21]

[20] Since $\hat{\theta}$ is a forecast on the "household side" it seems natural to assume $\theta_y = \hat{\theta}$, and it is plausible to consider cases where $\theta_\pi = \theta_y$.

[21] In the NE panel, there is a small unlabeled region along the horizontal axis and bounded above by a thin, downward-sloping line. In this region order two indeterminacy obtains.

Figure 2: Determinacy Properties: $\theta > 1$.

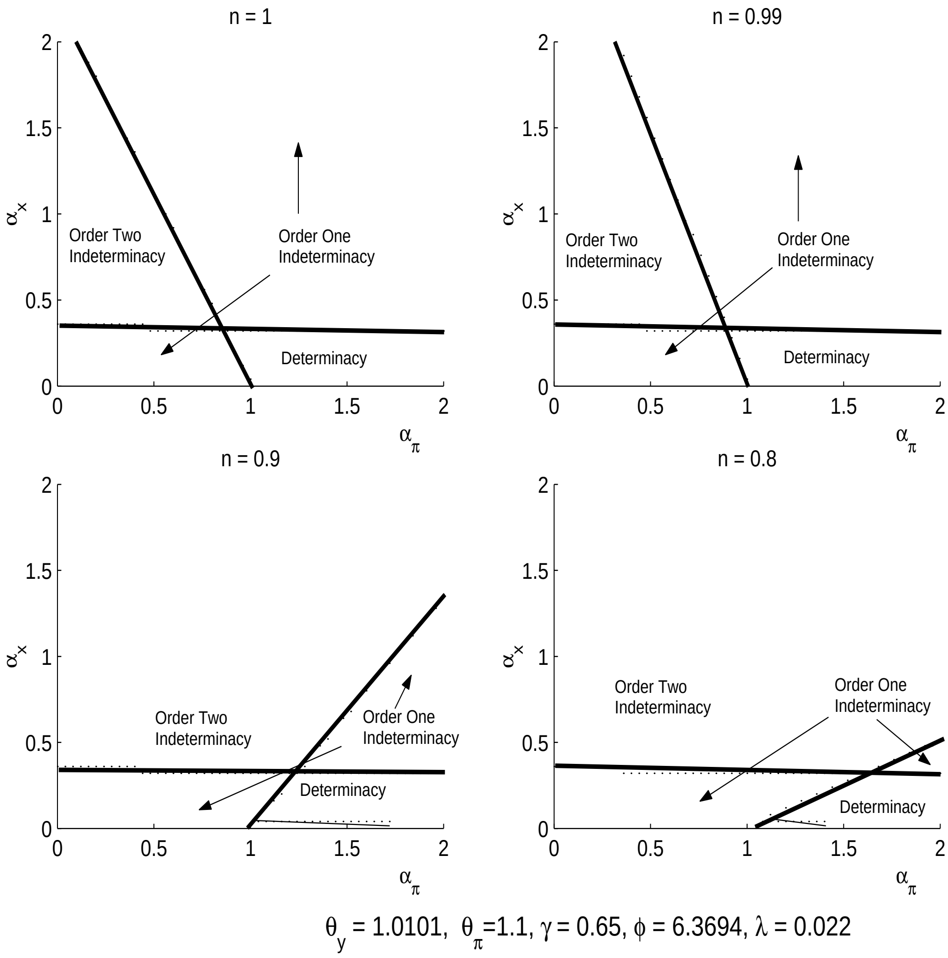


Whether heterogeneity is stabilizing or destabilizing, for a given fraction of rational agents, depends critically on the magnitude of the adaptation parameter θ. As an example, consider Figure 3, in which $\theta_\pi = 1.1$ and $\theta_y = \hat{\theta} = \beta^{-1} - \varepsilon$ as before. Here we are assuming that adaptive agents forecast π to grow at a faster rate than the present value of their bond holdings. We see that significant rotation obtains even if as few as 1% of private agents are adaptive, and if 10% are adaptive then indeterminacy obtains for somes rules which satisfy the Talyor principle and yield determinacy under rationality.[22] Moreover, the clockwise rotation of the determinacy frontier expands the region of order two indeterminacy.

Results 1 and 2 suggest that whether heterogeneity stabilizes or destabilizes depends on the distribution of agents across rational and adaptive expectations, and

[22]In the SW and SE panels, there are small unlabeled regions along the horizontal axis and bounded above by thin, downward-sloping lines. In these regions order two indeterminacy obtains.

Figure 3: Determinacy Properties: $\theta > 1$.



how strongly agents project past data in the adaptive predictor. When agents are adaptive in the more traditional sense, the region of determinacy may be more expansive. However, even slightly extrapolative or trend-chasing agents may destabilize a model that would be determinate under rationality. These results imply that a central bank uncertain about the precise nature of heterogeneity may desire policy robust for all reasonable forms of heterogeneity.

The intuition for these results is clear. The usual 'Taylor principle' intuition is that if nominal interest rates are not adjusted more than one for one with expected inflation then aggregate supply shocks (or self-fulfilling shocks to inflation expectations) will be further propagated through lower real rates leading to higher contemporaneous and future inflation. In the heterogeneous expectations case the degree of this future propagation depends on n and θ (i.e., how strongly adaptive agents extrapolate past data). In the case of $\theta < 1$, these adaptive beliefs are mean reverting and will not further propagate the shock into higher future inflation. Thus, a more tepid response in the nominal interest rate rule is consistent with stabilization. When $\theta > 1$, though, adaptive agents are trend-following and so inflation will be placed on a self-fulfilling path unless policy is particularly vigilant against inflation. Notice that this logic is consistent with policymakers' concerns that inflation expectations might become unhinged and out of the Fed's control.[23]

3.6 Discussion

The monetary policy literature, though wide and diverse, typically settles on the same recommendation: set policy so that the REE is determinate. At the heart of this recommendation is the property that a determinate steady state mitigates the potential for multiple equilibria to cause volatility of inflation and output. Our results indicate that a policy rule designed to implement determinacy may lead to inefficient outcomes if there exists heterogeneous expectations.[24]

To illustrate this point in the starkest terms, we parameterized the model so that it is determinate under rational expectations. We showed that with even a very small fraction of agents the determinacy properties may be very different. If agents extrapolate past data, then policy set to ensure a determinate REE may lead to indeterminacy with inefficient inflation and output volatility. Therefore, the results of this paper demonstrate that if policy attempts to achieve a determinate REE in a New Keynesian model and these heterogeneous expectations dynamics are present, the policy-maker may unwittingly destabilize the economy. As such, these results are a cautionary note and suggest that perhaps policy should guard against indeterminacy in heterogeneous expectations models.

Whether heterogeneity is stabilizing or destabilizing depends on the distribution and nature of the heterogeneity. If adaptive agents' expectations are dampening then policy does not have to be quite as vigilant against inflation. Instead, when adaptive agents extrapolate or 'trend-chase' then policy needs to be even more aggressive in its stance against inflation. These results are intuitive and seem to align with policymaker concerns such as Bernanke (2004) who emphasizes that adaptive beliefs becoming untethered from policy changes poses a significant challenge. However,

[23]That values of $\theta < 1$ tend to be stabilizing and $\theta > 1$ tend to indeterminacy and instability suggests a model where there is parameter learning and dynamic predictor selection. In this setting it would be interesting to see if in a calibrated version of the model, whether θ_t would tend to a number above 1 pushing the economy into the indeterminacy region. This would then add heterogeneous expectations as a potential explanation of the Great Inflation and Moderation. We leave such an examination to future work.

[24]It has also been emphasized that policy rules should be chosen so that the associated unique equilibrium is stable under learning: see for example, Bullard and Mitra (2002), Honkapohja and Mitra (2005) and Evans and McGough (2005). We note that the rule used to generate the plots in Figure 8 does produce an equilibrium that is stable under learning.

there are important open questions such as what specification of heterogeneous expectations is consistent with economic data. The benefit of our approach is that it is sufficiently general that it nests many forms of heterogeneity besides rational versus adaptive. However, the motivating example presented in this Section demonstrates that heterogeneity can strongly impact a New Keynesian model.

4 Conclusion

This paper derived a heterogeneous expectations version of a New Keynesian model from a micro-founded monetary economy with nominal rigidities. Importantly, the heterogeneous expectations model aggregates into a reduced form whose primary distinction from the representative agent model is that conditional expectations are replaced by a convex combination of expectation operators. Depending upon the beliefs of boundedly rational agents, the New Keynesian aggregate supply relation may depend on expectations of current and future inflation and output. As an example illustrating the potential implications of our approach, this paper also examined the impact of expectations heterogeneity on a model's determinacy properties. Our central findings are two-fold. Heterogeneity may be stabilizing or destabilizing, depending on the nature of the adaptive expectations mechanism. In case the mechanism is extrapolative, models which are determinate in case of rationality may be indeterminate and hence exhibit sunspot equilibria in the presence of even a small proportion of adaptive agents.

The theory presented here aims to bring more realism to a New Keynesian model while also demonstrating the conditions under which heterogeneous agent behavior aggregates into a simple reduced form model. Our primary theoretical result is to show that, under reasonable assumptions on beliefs, it is possible to represent a heterogeneous agent economy with an aggregate law of motion. From an empirical viewpoint, our theoretical analysis is important as recent studies have documented heterogeneous expectations and 'disagreement' of beliefs in survey data (e.g. Carroll (2003), Mankiw, Reis, and Wolfers (2003), and Branch (2004)). However, how these divergent beliefs might impact reduced form relations governing the evolution of the economy has been an open issue. We illustrate that aggregation is possible, and that the usual determinacy properties of the New Keynesian model may differ significantly. A question left open here is the extent to which a heterogeneous expectations New Keynesian model provides a good empirical fit of U.S. time-series.

These results can be linked to the growing call by economists for "robust policy." Many authors have argued that in the presence of model uncertainty, policy rules should be chosen to yield "good" outcomes across possible models, or, perhaps guard against particularly bad outcomes: see for example Levin and Williams (2003) and Brock, Durlauf and West (2005). Our results suggest that robustness against the impact of heterogeneity in expectations may also be warranted. Policy rules chosen to implement an equilibrium known to be unique under the assumption of rationality may result in indeterminacy. Our results are specific to the functional form of the expectations operator chosen by adaptive agents, and so, unless the form we analyze is taken quite seriously, we can offer little guidance as to how robust policy might be chosen; rather, our work points to the need for an improved theoretical and empirical understanding of heterogeneous expectations.

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