Market Price Mechanisms and Stackelberg General Equilibria*

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Abstract

This paper considers Stackelberg competition in a general equilibrium framework. The working of market power and the configurations of strategic interactions are complexified by the presence of an active leader. Two market price mechanisms are here studied: one is associated with the Stackelberg-Walras equilibrium and the other is linked to the Stackelberg-Cournot equilibrium. In the context of an exchange economy with a production sector, several results are obtained about equilibria mergings and about welfare comparisons.

1 Introduction

The concept of oligopoly equilibrium initially introduced by Stackelberg (1934) has mainly been developed in partial equilibrium analysis (see for instance Anderson and Engers (1992), Friedman (1992) and Tirole (1988)). This paper aims at extending the analysis of oligopolistic competition proposed by Stackelberg to a general equilibrium framework. We therefore propose to analyze the shapes and the consequences of market power under Stackelberg competition for an overall economy.

The concept of Stackelberg equilibrium is here modeled in the context of an exchange economy with a production sector. The simple model we develop is sufficient to display the diversity of behaviors and of strategic interactions associated with this complex form of competition. The "Stackelberg-Walras equilibrium" combines perfectly and imperfectly competitive behaviors, whereas the "Stackelberg-Cournot equilibrium" only involves strategic behaviors. In the former specification, the asymmetry takes place on the same side of the market and also between opposite sides of the market: some agents will act competitively as price-takers, while the other agents shall behave oligopolistically, with one leader

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and one follower. In the latter specification, only the first type of asymmetry remains, the exchange taking place between one leader and followers.

Stackelberg competition is here studied under two different market price mechanisms. One Stackelberg general equilibrium concept is therefore associated with each market price mechanism. The first one comes from a market equilibrium condition, with a market excess demand function for each commodity depending on a price system manipulated by the aggregate strategic supply. In this approach, the equilibrium market prices are determined for any given strategies (Walrasian step) and then the oligopolists' strategies are decided (Cournotian step). The Cournot-Walras equilibrium is the typical concept in this line initially developed by Codognato and Gabszewicz (1991), (1993), and later pursued by d'Aspremont *et alii* (1997) and by Gabszewicz and Michel (1997). The second market price mechanism is implemented in strategic market games proposed by Shapley and Shubik (1977) and developed by Amir *et alii* (1990)). Without money, the market relative price for each pair of commodities is the inverse ratio of the aggregate strategic supplies. These two kinds of market price mechanisms have been largely used in the literature devoted to general equilibrium under strategic interactions (see Gabszewicz (2002) and Giraud (2003)).

Tackling any Stackelberg general equilibrium concept, we assume the individual positions and the timing of moves as given, and therefore do not question the way a specific agent could or should become a leader (see Amir and Grilo (1999)). The purpose of this paper is rather the introduction of strategic interactions *à la* Stackelberg between interrelated markets. Correlatively, two interesting features can be put forward when casting Stackelberg competition into a general equilibrium framework. First, the market demand addressed to the producers is here made endogenous, which overcomes the lack of microfoundations that occurs with the usual assumption of an exogenous market demand function. Second, the model displays different kinds of heterogeneity and throws light on their consequences in terms of welfare. It especially integrates asymmetries across markets, which cannot be captured in partial equilibrium analyses[1].

Three types of results are obtained about mergings between equilibria and about production or welfare comparisons. For instance, when the productivity parameters of both oligopolists take the same value, the Stackelberg equilibria coincide with the Cournot equilibria for the two considered market mechanisms. And when these parameters take different values, there is no Pareto domination between the Stackelberg equilibria and the Cournot equilibria.

The paper is organized as follows. In section 2, we present the environment and define the economy. In section 3, we study the equilibrium allocation under the Stackelberg-Walras mechanism and establish two propositions. In section 4, we analyze the equilibrium allocation under the Cournot-Stackelberg mechanism and establish two propositions. In section 5, we compare the two kinds of Stackelberg allocations, and we introduce the competitive equilibrium in the general ranking of levels of production. In section 6, we conclude.

[1]The existence and the uniqueness analysis are beyond the scope of this paper. The existence of a general oligopoly equilibrium usually rises specific problems (see Bonnisseau and Florig (2003), Gabszewicz (2002)).

2 The economy

We consider an exchange economy with a production sector. It includes a finite set $\mathcal{H}$ of agents, indexed h, with $h = 1, ..., n+2$; and two divisible commodities (1 and 2). As an exchange economy, this framework displays agents 1 and 2 buying good 2 and the n other agents buying good 1 as consumption goods. As an economy with a production sector, this framework diplays agents 1 and 2 buying good 2 as an input, producing good 1 for self-consumption and for sale. The prices of the two goods expressed in a *numéraire* are denoted p_1 and p_2; and we may refer to the relative price $\frac{p_1}{p_2}$ as p. The preferences of any agent $h \in \mathcal{H}$ are represented by the following utility function:

$$U_h = \alpha \ln x_{h1} + (1-\alpha) \ln x_{h2} \text{ , with } \alpha \in (0,1), \forall h = 1, ..., n+2, \qquad (1)$$

where x_{h1} and x_{h2} represent the quantities of both goods consumed by any agent h. The structure of the initial endowments is assumed to be the same as in Gabszewicz and Michel (1997):

$$\omega_h = (0,0) \quad , \forall h = 1, 2, \qquad (2)$$

$$\omega_h = \left(0, \frac{1}{n}\right) \quad , \forall h = 3, ..., n+2. \qquad (3)$$

The amount of the second good is thus initially given and spread among the agents of the second type, each of them owning by assumption the same quantity $1/n$. The first good does not exist a priori in the economy, but can be produced by the first type of agents, who are endowed with a constant returns to scale technology[2]. Each linear technology tranforms the second commodity into the first one, according to the following specifications:

$$y_{11} = \frac{1}{\beta_1} z_{12} \text{ and } y_{21} = \frac{1}{\beta_2} z_{22} \text{ , with } \beta_1, \beta_2 > 0, \qquad (4)$$

where y_{11} and y_{21} are the amounts of commodity 1 that agents 1 and 2 can produce when using quantities z_{12} and z_{22} of good 2. So, the parameters β_1 and β_2 measure the productivities.

The two producing agents always behave strategically. When they compete together *à la* Stackelberg, the first agent is assumed to be the leader and the second is the follower. This asymmetry on the same side of the market could be based on a discrepancy in productivity: assuming $\beta_1 < \beta_2$, or $\frac{1}{\beta_1} > \frac{1}{\beta_2}$, would be a way to justify that the more productive agent is granted the leader position, while the less productive agent is granted the follower position. Under Stackelberg or Cournot competition, both oligopolists have two strategic

[2] As in Gabszewicz and Michel (1997), this specification is sufficient to capture some relevant features of strategic interactions in a general equilibrium framework with production.

decisions to make: which quantities y_{11} and y_{21} of good 1 to produce (which determines through (4) the amounts z_{12} and z_{22} of good 2 to be bought as inputs), and which amounts s_{11} and e_{21} of good 1 to supply in exchange for good 2 on the market. The strategy sets for the supplies of both oligopolists are respectively $S_1 = \{s_{11} \in \mathbb{R}_+ \mid 0 \leq s_{11} \leq y_{11}\}$ and $E_2 = \{e_{21} \in \mathbb{R}_+ \mid 0 \leq e_{21} \leq y_{21}\}$[3]. And their production sets are given by $Y_1 = \left\{(y_{11}, z_{12}) \in \mathbb{R}^2_+ \mid y_{11} \leq \frac{1}{\beta_1} z_{12}\right\}$ and by $Y_1 = \left\{(y_{21}, z_{22}) \in \mathbb{R}^2_+ \mid y_{21} \leq \frac{1}{\beta_2} z_{22}\right\}$. These strategy and production sets are thus convex. A pure strategy for oligopolist 1 is a pair (y_{11}, s_{11}) ,with $y_{11} \in Y_1$, $s_{11} \in S_1$; and a pure strategy for oligopolist 2 is a pair (e_{21}, y_{21}), with $y_{21} \in Y_2$, $e_{21} \in E_2$. Hence, the profit of each oligopolist writes respectively $\Pi_1 (s_{11}, y_{11}) = p_1 s_{11} - p_2 \beta_1 y_{11}$ and $\Pi_2 (e_{21}, y_{21}) = p_1 e_{21} - p_2 \beta_2 y_{21}$. The allocations (x_{h1}, x_{h2}), $\forall h = 1, 2$, resulting of the market process follow. Agent 1 obtains in exchange of s_{11} a quantity $\frac{p_1}{p_2} s_{11}$ of good 2, and finally consumes an amount $x_{12} = \frac{\Pi_1}{p_2}$ of this good and a quantity $x_{11} = y_{11} - s_{11}$ of good 1. Similarly, agent 2 obtains in exchange of e_{21} a quantity $\frac{p_1}{p_2} e_{21}$ of good 2, and finally consumes an amount $x_{22} = \frac{\Pi_2}{p_2}$ of this good and a quantity $x_{21} = y_{21} - e_{21}$ of good 1.

The n agents endowed in good 2 behave either competitively or strategically. When acting competitively, these agents have only one decision to make: which quantities of the two goods (x_{h1}, x_{h2}) they want to consume, taking the price system (p_1, p_2) as given. Denoting E_h the strategic set of any agent h, $h = 3, ..., n + 2$, we have $E_h = \{\varnothing\}$. In this case, the allocation (x_{h1}, x_{h2}) of agent h satisfies $p_1 x_{h1} + p_2 x_{h2} \leq \frac{p_2}{n}$, $\forall h = 3, ..., n + 2$. When acting strategically, they will try to manipulate the price system (p_1, p_2) in order to obtain a more favorable rate of exchange. Let us denote e_{h2} the pure strategy of agent h, $h = 3, ..., n{+}2$, with $e_{h2} \in \left[0, \frac{1}{n}\right]$. The strategic set of any agent h, $h = 3, ..., n{+}2$, is then $S_h = \left\{e_{h2} \in \mathbb{R}_+ \mid 0 \leq e_{h2} \leq \frac{1}{n}\right\}$. Any h will thus obtain in exchange of e_{h2} a quantity $\frac{p_2}{p_1} e_{h2}$ of good 1. In that case, the market process leads to the final allocation (x_{h1}, x_{h2}) such that $x_{h1} = \frac{p_2}{p_1} e_{h2}$ and $x_{h2} = \frac{1}{n} - e_{h2}$.

Beyond the three shapes of behavior at stake, two kinds of market price mechanism will be considered. The first one is basically a market equilibrium condition, defined for given strategies; while the second one insures the turn over of the supplied goods, the relative price being the inverse ratio of brought quantities. We here introduce a leader *à la* Stackelberg under both market price mechanisms in order to analyze the strategic interactions and to determine the associated equilibrium allocations.

Let us now define an economy for this environment[4].

Definition 1 *An economy is a collection of agents, endowments, production and strategic sets* $\xi = \{(\omega_1, Y_1, S_1), (\omega_2, Y_2, E_2), (\omega_h, E_h)\}_{h=3,...,n+2}$.

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[3] We will denote the supply strategy of agent 1 by s_{11} when s/he acts as a Stackelberg leader and by e_{11} when s/he acts as a Cournot oligopolist.

[4] We can notice that for $\alpha = 0$ the economy is autarkic; and for $\alpha = 1$ commodity 2 is a pure input and agents only consume good 1.

For this economy, we first study the Stackelberg-Walras equilibrium (SWE), for which $E_h = \{\varnothing\}$, $\forall h = 3, ..., n+2$. We then relax the price-taking assumption and study the Stackelberg-Cournot equilibrium (SCE), for which $E_h = \left\{e_{h2} \in \mathbb{R}_+ \mid 0 \leq e_{h2} \leq \frac{1}{n}\right\}$, $\forall h = 3, ..., n+2$. We finally compare these two equilibria to the Cournot-Walras equilibrium (CWE) and to the Cournot equilibrium (CE).

3 The Stackelberg-Walras equilibrium

In the SWE framework, it is considered that agents having endowments in good 2 act competitively, whereas the other agents behave strategically. The leader manipulates the follower's decision and all these oligopolists manipulate the price by restricting their supply, while the many other agents are price-takers.

A SWE for the economy ξ is a non cooperative equilibrium of a game where the players are the oligopolists, the strategies are their production and supply decisions and the payoffs are their utility levels.

Definition 2 *A SWE for the economy ξ is given by a market price $\tilde{p}(\tilde{s}_{11}, \tilde{e}_{21})$, a pair of strategies $(\tilde{s}_{11}, \tilde{e}_{21})$, with $\tilde{s}_{11} \in [0, \tilde{y}_{11}]$ and $\tilde{e}_{21} \in [0, \tilde{y}_{21}]$, and an allocation $(\tilde{x}_1, ..., \tilde{x}_h, ..., \tilde{x}_{n+2}) \in \mathbb{R}_+^{2(n+2)}$ such that: (i) $\tilde{x}_h = x_h(\tilde{s}_{11}, \tilde{e}_{21}, \tilde{p})$, $\forall h$, (ii) $Max U_h(x_h)$ s.t. $px_h \leq p_2 \omega_{h2}$, $\forall h = 3, ..., n+2$, (iii) $s_{11} + e_{21} = \sum_{h=3}^{h=n+2} x_{h1}(p)$, $\forall s_{11}$, $\forall e_{21}$, (iv) $U_2(\tilde{x}_2(\tilde{s}_{11}, \tilde{e}_{21}, \tilde{p})) \geq U_2(x_2(\tilde{s}_{11}, e_{21}, p))$, $\forall e_{21}$ and (v) $U_1(\tilde{x}_1(\tilde{s}_{11}, \tilde{e}_{21}, \tilde{p})) \geq U_1(x_1(s_{11}, e_{21}(s_{11}), p))$, $\forall s_{11}$.*

The Stackelberg-Walras allocations depend on competitive and strategic decisions. This equilibrium concept can be viewed as a subgame perfect equilibrium of a three-stage game[5]. In the first step the competitive equilibrium is computed for any value of each strategy. In the second step the follower's reaction functions are determined. In the third step the leader's optimal decisions are made. The story is solved by backward induction, considering first all the competitive behaviors, then the decisions of the follower and finally the choices of the leader.

Therefore, each agent h, $h = 3, ..., n+2$, solves the following maximization program $\underset{(x_{h1}, x_{h2})}{Max}\ \alpha \ln x_{h1} + (1-\alpha) \ln x_{h2}$ s.t. $p_1 x_{h1} + p_2 x_{h2} \leq \frac{p_2}{n}$. This leads notably to the vector of competitive demand functions for commodities 1 and 2, i.e. $x_h(p_1, p_2) = \left(\frac{\alpha}{n}\frac{p_2}{p_1}, \frac{1-\alpha}{n}\right)$, $\forall h = 3, ..., n+2$. The aggregate demand function in good 1 which is addressed to the oligopolists is thus $x_1 = \alpha \frac{p_2}{p_1}$. The market-clearing condition for good 1 then writes $\alpha \frac{p_2}{p_1} = s_{11} + e_{21}$, which leads to:

$$\frac{p_1}{p_2} = \frac{\alpha}{s_{11} + e_{21}}. \tag{5}$$

[5]The concept of subgame perfect equilibrium of a two stage game was initially proposed for the Cournot-Walras equilibrium by Busetto *et alii* (2008).

Equation (5) verifies $\frac{\partial\left(\frac{p_1}{p_2}\right)}{\partial\alpha} > 0$: the relative price depends on the structure of preferences. And $\frac{\partial\left(\frac{p_1}{p_2}\right)}{\partial(s_{11}+e_{21})} < 0$: the oligopolists can get a better price restricting their supplies.

The strategic plan of the follower is determined by two elements: s/he manipulates the market price and s/he takes the leader's strategy as given. The objective function of the follower can be written as the payoff function $V_2(e_{21}, y_{21}) = \alpha \ln(y_{21} - e_{21}) + (1-\alpha)\ln\left(\frac{p_1}{p_2}e_{21} - \beta_2 y_{21}\right)$ (see Appendix 1). The follower maximizes $V_2(e_{21}, y_{21})$ with respect to e_{21} and y_{21}:

$$\underset{\{e_{21},y_{21}\}}{Arg\max}\ \alpha \ln(y_{21} - e_{21}) + (1-\alpha)\ln\left(\frac{\alpha}{s_{11}+e_{21}}e_{21} - \beta_2 y_{21}\right). \tag{6}$$

The optimality conditions with respect to e_{21} and y_{21}, i.e. $\partial V_2/\partial e_{21} = 0$ and $\partial V_2/\partial y_{21} = 0$, yield:

$$\frac{\alpha}{y_{21} - e_{21}} = \frac{\alpha(1-\alpha)\frac{s_{11}}{(s_{11}+e_{21})^2}}{\frac{\alpha}{s_{11}+e_{21}}e_{21} - \beta_2 y_{21}} = \frac{(1-\alpha)\beta_2}{\frac{\alpha}{s_{11}+e_{21}}e_{21} - \beta_2 y_{21}}\ , \tag{7}$$

From (7) we can deduce the two reaction functions of the follower:

$$e_{21}(s_{11}) = -s_{11} + \sqrt{\frac{\alpha}{\beta_2}s_{11}}\ , \tag{8}$$

$$y_{21}(s_{11}) = \frac{\alpha^2}{\beta_2} + (1-2\alpha)\sqrt{\frac{\alpha}{\beta_2}s_{11}} - (1-\alpha)s_{11}. \tag{9}$$

Since s_{11} is the only leader's decision influencing the relative price, these two reaction functions depend on s_{11} and not on y_{11}. These functions inherit the properties of the relations given in (1) and (4) and are all continuous. We must have $y_{21} > e_{21} > 0$. Indeed, $e_{21} > 0$ is obtained when $s_{11} < \frac{\alpha}{\beta_2}$ (see Equation (12)), i.e. $2\beta_1 > \beta_2$, which is assumed to be verified. And $y_{21} > e_{21}$ (which guarantees that $y_{21} > 0$) if $\frac{\beta_1}{\beta_2} + \frac{1}{4}\frac{\beta_2}{\beta 1} > 1$. This inequality stands if and only if $\varphi\left(\frac{\beta_1}{\beta_2}\right) > 0$, with $\varphi \equiv \left(\frac{\beta_1}{\beta_2}\right)^2 - \frac{\beta_1}{\beta_2} + \frac{1}{4}$. We easily verify that $\varphi \geq 0$ on $\left(\frac{1}{2}, 1\right)$. Additionally, $\frac{\partial e_{21}}{\partial s_{11}} > 0$ for $s_{11} \in \left[0, \frac{\alpha}{4\beta_2}\right]$ and $\frac{\partial e_{21}}{\partial s_{11}} < 0$ for $s_{11} \in \left[\frac{\alpha}{4\beta_2}, \frac{\alpha}{\beta_2}\right]$[6]. We also have $\frac{\partial^2 e_{21}}{\partial s_{11}^2} < 0$. The same last three properties hold for $y_{21}(s_{11})$. Finally, $\frac{\partial e_{21}}{\partial \beta_2} < 0$ and $\frac{\partial e_{21}}{\partial \alpha} > 0$: the strategic supply of the follower increases when her/his productivity goes up and depends on the structure of preferences.

The strategic plan of the leader is determined by two elements: the leader manipulates the market price, as given by (5), and the follower strategy e_{21}. Using the same argument as previously, one obtains the following payoff function

[6] The equilibrium strategies in (12) will show that $\tilde{s}_{11} \in \left[\frac{\alpha}{4\beta_2}, \frac{\alpha}{\beta_2}\right]$, where strategic substituabilities occur. Conversely, strategic complementarities are displayed for $s_{11} \in \left[0, \frac{\alpha}{4\beta_2}\right]$.

for the leader: $V_1(s_{11}, y_{11}) = \alpha \ln(y_{11} - s_{11}) + (1-\alpha)\ln\left(\frac{p_1}{p_2}s_{11} - \beta_1 y_{11}\right)$. This function has to be maximized with respect to s_{11} and y_{11}:

$$\underset{\{\tilde{s}_{11},\tilde{y}_{11}\}}{Arg\max}\ \alpha \ln(y_{11} - s_{11}) + (1-\alpha)\ln\left(\frac{\alpha}{s_{11} + e_{21}(s_{11})}s_{11} - \beta_1 y_{11}\right), \tag{10}$$

The optimality conditions with respect to s_{11} and y_{11}, i.e. $\partial V_2/\partial s_{11} = 0$ and $\partial V_2/\partial y_{11} = 0$, yield:

$$\frac{\alpha}{y_{21} - e_{21}(s_{11})} = \frac{\alpha(1-\alpha)\frac{e_{21}(s_{11})}{(s_{11}+e_{21}(s_{11}))^2}}{\Delta} = \frac{(1-\alpha)\beta_2}{\Delta}\ , \tag{11}$$

where $\Delta \equiv \frac{\alpha}{s_{11}+e_{21}(s_{11})}e_{21}(s_{11}) - \beta_2 y_{21}$. This gives the vector of equilibrium strategies of the leader:

$$(\tilde{s}_{11}, \tilde{y}_{11}) = \left(\frac{\alpha}{4}\frac{\beta_2}{\beta_1^2}, \frac{\alpha(1+\alpha)}{4}\frac{\beta_2}{\beta_1^2}\right). \tag{12}$$

We obviously have $(\tilde{s}_{11}, \tilde{y}_{11}) \in \mathbb{R}^2_{++}$, and $\tilde{y}_{11} > \tilde{s}_{11}$ as $\alpha > 0$.

We deduce the equilibrium follower's equilibrium strategies $\tilde{e}_{21}(\tilde{s}_{11})$ and $\tilde{y}_{21}(\tilde{s}_{11})$:

$$(\tilde{e}_{21}, \tilde{y}_{21}) = \left(\frac{\alpha(2\beta_1 - \beta_2)}{4\beta_1^2}, \frac{\alpha^2}{\beta_2} + \frac{\alpha(1-2\alpha)}{2\beta_1} - \frac{\alpha(1-\alpha)\beta_2}{4\beta_1^2}\right). \tag{13}$$

From (5), (12) and (13), the equilibrium relative price is:

$$\left(\frac{\tilde{p}_1}{p_2}\right) = 2\beta_1. \tag{14}$$

We have $\frac{\partial\left(\frac{\tilde{p}_1}{p_2}\right)}{\partial\beta_1} > 0$. As the leader's productivity increases, commodity 1 becomes less valued relatively to commodity 2.

The equilibrium allocations are:

$$(\tilde{x}_{11}, \tilde{x}_{12}) = \left(\left(\frac{\alpha}{2}\right)^2\frac{\beta_2}{\beta_1^2}, \frac{\alpha(1-\alpha)}{4}\frac{\beta_2}{\beta_1}\right), \tag{15}$$

$$(\tilde{x}_{21}, \tilde{x}_{22}) = \left(\frac{\alpha^2}{\beta_2}\left(\frac{2\beta_1 - \beta_2}{2\beta_1}\right)^2, \alpha(1-\alpha)\left(\frac{2\beta_1 - \beta_2}{2\beta_1}\right)^2\right), \tag{16}$$

$$(\tilde{x}_{h1}, \tilde{x}_{h2}) = \left(\frac{1}{2\beta_1}\frac{\alpha}{n}, \frac{1-\alpha}{n}\right),\ \forall h = 3, ..., n+2. \tag{17}$$

The corresponding utility levels are:

$$\tilde{U}_1 = \Lambda(\alpha) + \ln\beta_2 - (1+\alpha)\ln\beta_1 - 2\ln 2, \tag{18}$$

$$\tilde{U}_2 = \Lambda(\alpha) + 2[\ln(2\beta_1 - \beta_2) - \ln 2 - \ln \beta_1] - \alpha \ln \beta_2, \tag{19}$$

$$\tilde{U}_h = (1-\alpha)\ln(1-\alpha) + \alpha(\ln \alpha - \ln 2 - \ln \beta_1) - \ln n,\ \forall h \neq 1, 2, \tag{20}$$

where $\Lambda(\alpha) \equiv (1+\alpha)\ln\alpha + (1-\alpha)\ln(1-\alpha)$.

It can be noticed that the utility of each of the two oligopolists depends positively on his/her productivity and negatively on the other's productivity[7]. The welfare of each competitive agent decreases with their number but increases with the leader's productivity.

It is meaningful to compare the market outcomes holding under the SWE with these that would prevail if both producers played a Cournot game. This is condensed in the two following propositions.

Proposition 3 *When $\beta_1 = \beta_2$, the Stackelberg-Walras equilibrium coincides with the Cournot-Walras equilibrium.*

Proof. *We verify that the equilibrium strategies of both oligopolists, the market price and the SWE allocations reached for $\beta_1 = \beta_2$ correspond to the equilibrium strategies, the market price and the allocations that would be obtained in an environment where both producers would play a Cournot game, all the consumers owning good 2 still behaving competitively. A Cournot-Walras equilibrium for the economy ξ is given by a market price $\breve{p}(\breve{e}_{11}, \breve{e}_{21})$, a pair of strategies $(\breve{e}_{11}, \breve{e}_{21})$ and an allocation $(\breve{x}_1, ..., \breve{x}_h, ..., \breve{x}_{n+2}) \in \mathbb{R}_+^{2(n+2)}$ such that: (i) $\breve{x}_h = x_h(\breve{e}_{11}, \breve{e}_{21}, \breve{p})$, $\forall h$, (ii) $Max U_h(\breve{x}_h)$ s.t. $p\breve{x}_h \leq p_2\omega_{h2}$, $h = 3, ..., n+2$, (iii) $e_{11} + e_{21} = \sum_h x_{h1}(p)$, $h \neq 1, 2$, $\forall e_{11}, \forall e_{21}$, (iv) $U_2(\breve{x}_2(\breve{e}_{11}, \breve{e}_{21}, \breve{p})) \geq U_2(x_2(\breve{e}_{11}, e_{21}, p))$, $\forall e_{21}$, and (v) $U_1(\breve{x}_1(\breve{e}_{11}, \breve{e}_{21}, \breve{p})) \geq U_1(x_1(e_{11}, \breve{e}_{21}, p))$, $\forall e_{11}$. Consider the maximization programs as given by (6) and by (10) with $s_{11} \equiv e_{11}$ and $e_{21}(s_{11}) \equiv e_{21}$. The equilibrium strategies $(\breve{e}_{11}, \breve{e}_{21})$ are solutions of the two equations $e_{11} = -e_{21} + \sqrt{\frac{\alpha e_{21}}{\beta_1}}$ and $e_{21} = -e_{11} + \sqrt{\frac{\alpha e_{11}}{\beta_2}}$. This leads to the following pair of equilibrium strategies: $(\breve{e}_{11}, \breve{e}_{21}) = \left(\frac{\alpha\beta_2}{(\beta_1+\beta_2)^2}, \frac{\alpha\beta_1}{(\beta_1+\beta_2)^2}\right)$. The equilibrium price is then $\breve{p} = \beta_1 + \beta_2$. The production strategies are thus $(\breve{y}_{11}, \breve{y}_{21}) = \left(\alpha\frac{\beta_2}{\beta_1}\frac{(\beta_1+\alpha\beta_2)}{(\beta_1+\beta_2)^2}, \alpha\frac{\beta_1}{\beta_2}\frac{(\alpha\beta_1+\beta_2)}{(\beta_1+\beta_2)^2}\right)$. The CWE is given by $(\breve{x}_{11}, \breve{x}_{12}) = \left(\frac{1}{\beta_1}\left(\frac{\alpha\beta_2}{\beta_1+\beta_2}\right)^2, \alpha(1-\alpha)\left(\frac{\beta_2}{\beta_1+\beta_2}\right)^2\right)$, $(\breve{x}_{21}, \breve{x}_{22}) = \left(\frac{1}{\beta_2}\left(\frac{\alpha\beta_1}{\beta_1+\beta_2}\right)^2, \alpha(1-\alpha)\left(\frac{\beta_1}{\beta_1+\beta_2}\right)^2\right)$ and for $h = 3, ..., n+2$, $(\breve{x}_{h1}, \breve{x}_{h2}) = \left(\frac{1}{(\beta_1+\beta_2)}\frac{\alpha}{n}, \frac{1-\alpha}{n}\right)$. When $\beta_1 = \beta_2 = \beta$, (12)-(14) lead to $(\tilde{s}_{11}, \tilde{e}_{21}) = \left(\frac{\alpha}{4\beta}, \frac{\alpha}{4\beta}\right)$ and $\tilde{p} = 2\beta$; and (15)-(17) yield $(\tilde{x}_{11}, \tilde{x}_{12}) = \left(\frac{1}{\beta}\left(\frac{\alpha}{2}\right)^2, \frac{\alpha(1-\alpha)}{4}\right)$, $(\tilde{x}_{21}, \tilde{x}_{22}) = \left(\frac{1}{\beta}\left(\frac{\alpha}{2}\right)^2, \frac{\alpha(1-\alpha)}{4}\right)$, $(\tilde{x}_{h1}, \tilde{x}_{h2}) = \left(\frac{1}{2\beta}\frac{\alpha}{n}, \frac{1-\alpha}{n}\right)$, $\forall h = 3, ..., n+2$. For $\beta_1 = \beta_2 = \beta$, we finally verify that $(\breve{e}_{11}, \breve{e}_{21}) = (\tilde{s}_{11}, \tilde{e}_{21})$ and $\breve{p} = \tilde{p}$, which leads to $(\breve{x}_{h1}, \breve{x}_{h2}) = (\tilde{x}_{h1}, \tilde{x}_{h2})$, $\forall h$. QED.* ■

[7] The utility $\tilde{U}_2$ can be written $\tilde{U}_2 = \Lambda(\alpha) - 2\ln 2 + 2\ln\left(2 - \frac{\beta_2}{\beta_1}\right) - \alpha\ln\beta_2$.

Switching from follower to leader, agent 1 does not modify anything when her/his leading position is not backed up by an objective competitive advantage (such as a better productivity) compared to agent 2.

Proposition 4 *When $\beta_1 < \beta_2$, the Cournot-Walras equilibrium is not Pareto dominated by the Stackelberg-Walras equilibrium.*

Proof. *The utility levels reached by the two Cournotian oligopolists and the competitive agents are respectively $\breve{U}_1 = \Lambda(\alpha) + 2\ln\beta_2 - \alpha\ln\beta_1 - 2\ln(\beta_1+\beta_2)$, $\breve{U}_2 = \Lambda(\alpha) + 2\ln\beta_1 - \alpha\ln\beta_2 - 2\ln(\beta_1+\beta_2)$ and $\breve{U}_h = \Lambda(\alpha) - \ln\alpha - \ln n - \alpha\ln(\beta_1+\beta_2)$, $\forall h = 3,...,n+2$. For the first oligopolist, we have $\tilde{U}_1 - \breve{U}_1 = 2\ln(\beta_1+\beta_2) - \ln\beta_1 - \ln\beta_2 - 2\ln 2 > 0$ since $(\beta_1-\beta_2)^2 > 0$. For the other oligopolist, we have $\tilde{U}_2 - \breve{U}_2 = 2\ln(\beta_1+\beta_2) + 2\ln(2\beta_1-\beta_2) - 4\ln\beta_1 - 2\ln 2 < 0$. For all the other agents, this yields $\tilde{U}_h - \breve{U}_h = \alpha\ln(\beta_1+\beta_2) - \alpha\ln\beta_1 - \alpha\ln 2 > 0$. QED.* ■

When agent 1 benefits from a competitive advantage ($\beta_1 < \beta_2$), s/he does better as a leader than as a follower; and agent 2 is better off confronted to a Cournot competitor than to a Stackelberg one. Moreover, the welfare of agents endowed with good 2 is higher when they face a Stackelberg competition and not a Cournot one.

4 The Stackelberg-Cournot equilibrium

In the SCE framework, it is considered that all agents behave strategically, with agent 1 as the leader. The set of followers now also encompasses all the agents endowed with good 2. The pure strategy of agent h, $h = 3,...,n+2$, is e_{h2} and h obtains in exchange of e_{h2} a quantity $\frac{p_2}{p_1}e_{h2}$ of good 1.

Definition 5 *A SCE for ξ is given by a market price $\hat{p}(\hat{s}_{11},\hat{e}_{21},\hat{e}_{h2})$, a $(n+2)$-tuple of strategies $(\hat{s}_{11},\hat{e}_{21},\hat{e}_{h2})$, with $\hat{s}_{11} \in [0,\hat{y}_{11}]$, $\hat{e}_{21} \in [0,\hat{y}_{21}]$ and $\hat{e}_{h2} \in [0,\frac{1}{n}]$, and an allocation $(\hat{x}_1,\hat{x}_2,...,\hat{x}_h,...,\hat{x}_{n+2}) \in \mathbb{R}_+^{2(n+2)}$ such that: (i) $\hat{x}_h = x_h(\hat{s}_{11},\hat{e}_{21},\hat{e}_{i2},\hat{p})$, $\forall h$, $\forall i = 3,...,n+2$, (ii) $s_{11} + e_{21} = \frac{1}{p}\sum_h e_{h2}$, $h \neq 1,2$, $\forall s_{11}$, $\forall e_{21}$ and $\forall e_{h2}$ (iii) $U_h(\hat{x}_h(\hat{s}_{11},\hat{e}_{21},\hat{e}_{h2},\hat{e}_{-h2}) \geq U_h(x_h(\hat{s}_{11},\hat{e}_{21},e_{h2},\hat{e}_{-h2}))$, $\forall e_{h2}$, $\forall h = 3,...,n+2$, (iv) $U_2(\hat{x}_2(\hat{s}_{11},\hat{e}_{21},\ \hat{e}_{h2}) \geq U_2(x_2(\hat{s}_{11},e_{21},\hat{e}_{h2}))$, $\forall e_{21}$, $\forall h = 3,...,n+2$ and (v) $U_1(\hat{x}_1(\hat{s}_{11},\hat{e}_{21},\ \hat{e}_{h2}) \geq U_1(x_1(s_{11},e_{21}(s_{11}),e_{h2}(s_{11}))$, $\forall s_{11}$, $\forall h \neq 1,2$.*

This equilibrium concept can be viewed as the subgame perfect equilibrium of a two-stage game. In the first step all the reaction functions are determined; in the second step the leader's optimal decision is made. The story is solved by backward induction, considering first all the strategic reactive decisions, and then the strategic active choice of the leader.

The market price which insures market-clearing is given by:

$$\frac{p_1}{p_2} = \frac{\sum_{h=3}^{h=n+2} e_{h2}}{s_{11} + e_{21}} \ . \tag{21}$$

Taking the $(n-1)e_{-h2}$, s_{11} and e_{21} strategies as given, each oligopolist h, $h=3,...,n+2$, maximizes her/his utility. Following a procedure similar to the argument given in section 3, it can be shown that the strategy set can be restricted to $\bar{E}_h$, with $\bar{E}_h = \left\{ e_{h2} \in \mathbb{R}_+ \mid 0 \leq e_{h2} \leq \frac{\alpha}{n} \right\}$, $\forall h=3,...,n+2$. Consequently, the utility level reached by any oligopolist h, $h=3,...,n+2$ is given by $V_h = \alpha \ln \left(\frac{p_2}{p_1} e_{h2} \right) + (1-\alpha) \ln \left(\frac{1}{n} - e_{h2} \right)$. Then:

$$\underset{\{e_{h2}\}}{Arg \max}\ \alpha \ln \left(\frac{(s_{11}+e_{21})}{e_{h2}+(n-1)e_{-h2}} e_{h2} \right) + (1-\alpha) \ln \left(\frac{1}{n} - e_{h2} \right), \tag{22}$$

which gives the following reaction functions:

$$e_{h2} = \frac{\alpha}{n}\gamma,\ \forall h=3,...,n+2, \tag{23}$$

where $\gamma \equiv \frac{n-1}{n-\alpha}$, with $\frac{1}{2} \leq \gamma \leq 1$.

Taking the strategy s_{11} and the n strategies e_{h2} as given, the Cournotian follower who produces good 1 solves the program:

$$\underset{\{e_{21},y_{21}\}}{Arg \max}\ \alpha \ln (y_{21} - e_{21}) + (1-\alpha) \ln \left(\frac{\alpha\gamma}{s_{11}+e_{21}} e_{21} - \beta_2 y_{21} \right), \tag{24}$$

which gives the reaction functions:

$$e_{21}(s_{11}) = -s_{11} + \sqrt{\frac{\alpha}{\beta_2}\gamma s_{11}}. \tag{25}$$

$$y_{21}(s_{11}) = \frac{\alpha^2}{\beta_2}\gamma + (1-2\alpha)\sqrt{\frac{\alpha}{\beta_2}\gamma s_{11}} - (1-\alpha)s_{11}. \tag{26}$$

As for (8)-(9), these two reaction functions depend on s_{11} and not on y_{11}. We can verify that $y_{21} > e_{21} > 0$. Additionally, $\frac{\partial e_{21}}{\partial s_{11}} > 0$ for $s_{11} \in \left[0, \frac{\alpha\gamma}{4\beta_2}\right]$ and $\frac{\partial e_{21}}{\partial s_{11}} < 0$ for $s_{11} \in \left[\frac{\alpha\gamma}{4\beta_2}, \frac{\alpha\gamma}{\beta_2}\right]$. We also have $\frac{\partial^2 e_{21}}{\partial s_{11}^2} < 0$. The same last three properties hold for $y_{21}(s_{11})$. Finally, $\frac{\partial e_{21}}{\partial \beta_2} < 0$ and $\frac{\partial e_{21}}{\partial \alpha} > 0$: again, the strategic supply of the follower increases when her/his productivity goes up and depends on the structure of preferences.

Considering the best responses of all the followers, the leader maximizes her/his utility:

$$\underset{\{\hat{s}_{11},\hat{y}_{11}\}}{Arg \max} \alpha \ln (y_{11} - s_{11}) + (1-\alpha) \ln \left(\sqrt{\alpha\beta_2\gamma s_{11}} - \beta_1 y_{11} \right), \tag{27}$$

which gives the vector of equilibrium strategies of the leader:

$$(\hat{s}_{11}, \hat{y}_{11}) = \left(\frac{\alpha}{4}\frac{\beta_2}{\beta_1^2}\gamma, \frac{\alpha(1+\alpha)}{4}\frac{\beta_2}{\beta_1^2}\gamma \right). \tag{28}$$

We deduce the followers' strategies $\hat{e}_{21}(\hat{s}_{11})$, $\hat{y}_{21}(\hat{s}_{11})$ and $\hat{e}_{h2}$:

$$(\hat{e}_{21},\hat{y}_{21})=\left(\frac{\alpha(2\beta_1-\beta_2)}{4\beta_1^2}\gamma,\frac{\alpha^2}{\beta_2}\gamma+\frac{\alpha(1-2\alpha)}{2\beta_1}\gamma-\frac{\alpha(1-\alpha)\beta_2}{4\beta_1^2}\gamma\right), \tag{29}$$

$$\hat{e}_{h2}=\frac{\alpha}{n}\gamma, \forall h=3,...,n+2. \tag{30}$$

The equilibrium price is then:

$$\left(\frac{\hat{p}_1}{p_2}\right)=2\beta_1. \tag{31}$$

The equilibrium allocations are:

$$(\hat{x}_{11},\hat{x}_{12})=\left(\left(\frac{\alpha}{2}\right)^2\frac{\beta_2}{\beta_1^2}\gamma,\frac{\alpha(1-\alpha)}{4}\frac{\beta_2}{\beta_1}\gamma\right), \tag{32}$$

$$(\hat{x}_{21},\hat{x}_{22})=\left(\frac{\alpha^2}{\beta_2}\left(\frac{2\beta_1-\beta_2}{2\beta_1}\right)^2\gamma,\alpha(1-\alpha)\left(\frac{2\beta_1-\beta_2}{2\beta_1}\right)^2\gamma\right), \tag{33}$$

$$(\hat{x}_{h1},\hat{x}_{h2})=\left(\frac{1}{2\beta_1}\frac{\alpha}{n}\gamma,\frac{1-\alpha\gamma}{n}\right), \forall h=3,...,n+2. \tag{34}$$

The corresponding utility levels are:

$$\hat{U}_1=\tilde{U}_1+\ln\gamma, \tag{35}$$

$$\hat{U}_2=\tilde{U}_2+\ln\gamma, \tag{36}$$

$$\hat{U}_h=\tilde{U}_h+\alpha\ln\gamma+(1-\alpha)\ln\left(\frac{1-\alpha\gamma}{1-\alpha}\right), \forall h=3,...,n+2. \tag{37}$$

Proposition 6 *When $\beta_1=\beta_2$, the Stackelberg-Cournot equilibrium coincides with the Cournot equilibrium.*

Proof. We verify that the equilibrium strategies of both oligopolists, the market price and the SCE allocations reached for $\beta_1=\beta_2$ correspond to the equilibrium values that would be obtained in an environment where all agents would play a Cournot game. A Cournot equilibrium for the economy ξ is given by a market price $\check{p}(\check{e}_{h\cdot}), \forall h$, a vector of strategies $(\check{e}_{11},\check{e}_{21},...,\check{e}_{h2},...,\check{e}_{hn+2})\in\mathbb{R}_+^{n+2}$ and an allocation $(\check{x}_1,...,\check{x}_h,...,\check{x}_{n+2})\in\mathbb{R}_+^{n+2}$ such that: (i) $\check{x}_h=x_h(\check{e}_{h\cdot},\check{p})$, $\forall h$, (ii) $e_{11}+e_{21}=\frac{1}{p}\sum_h e_{h2}$ and (iii) $U_h(\check{x}_h(\check{e}_{h\cdot},\check{e}_{-h\cdot},\check{p})\geq U_h(x_h(e_{h\cdot},\check{e}_{-h\cdot},p))$, $\forall h$. Consider (22) and (24). The equilibrium strategies for agent h, $h=3,...,n+2$ are $\check{e}_{h2}=\frac{\alpha}{n}\gamma$, while the equilibrium strategies for agents 1 and 2 $(\check{e}_{11},\check{e}_{21})=\left(\frac{\alpha\beta_2}{(\beta_1+\beta_2)^2}\gamma,\frac{\alpha\beta_1}{(\beta_1+\beta_2)^2}\gamma\right)$ are the solutions to $e_{11}=-e_{21}+\sqrt{\frac{\alpha}{\beta_1}\gamma e_{21}}$ and $e_{21}=-e_{11}+\sqrt{\frac{\alpha}{\beta_2}\gamma e_{11}}$. Then $\check{p}=\beta_1+\beta_2$. Additionally, $(\check{y}_{11},\check{y}_{21})=\left(\frac{\alpha(\beta_1+\alpha\beta_2)}{(\beta_1+\beta_2)^2}\frac{\beta_2}{\beta_1}\gamma,\frac{\alpha(\alpha\beta_1+\beta_2)}{(\beta_1+\beta_2)^2}\frac{\beta_1}{\beta_2}\gamma\right)$, from which we can deduce $(\check{x}_{11}, \check{x}_{12}) = \left(\frac{1}{\beta_1} \left(\frac{\alpha \beta_2}{\beta_1 + \beta_2} \right)^2 \gamma, \alpha(1-\alpha) \left(\frac{\beta_2}{\beta_1+\beta_2} \right)^2 \gamma \right)$ for the first producer, $(\check{x}_{21}, \check{x}_{22}) = \left(\frac{1}{\beta_2} \left(\frac{\alpha \beta_1}{\beta_1 + \beta_2} \right)^2 \gamma, \alpha(1-\alpha) \left(\frac{\beta_1}{\beta_1+\beta_2} \right)^2 \gamma \right)$ for the second one, and $(\check{x}_{h1}, \check{x}_{h2}) = \left(\frac{1}{(\beta_1+\beta_2)} \frac{\alpha}{n} \gamma, \frac{1-\alpha\gamma}{n} \right)$, $\forall h = 3, ..., n+2$. When $\beta_1 = \beta_2 = \beta$, (28)-(31) lead to $(\hat{e}_{11}, \hat{e}_{21}) = \left(\frac{\alpha}{4\beta}\gamma, \frac{\alpha}{4\beta}\gamma \right)$, $\forall h = 1, 2$, $\hat{e}_{h2} = \frac{\alpha}{n}\gamma$, $\forall h = 3, ..., n+2$ and $\hat{p} = 2\beta$. Moreover, (32)-(34) yield respectively $(\hat{x}_{11}, \hat{x}_{12}) = \left(\left(\frac{\alpha}{2}\right)^2 \frac{1}{\beta}\gamma, \frac{\alpha(1-\alpha)}{4}\gamma \right)$, $(\hat{x}_{21}, \hat{x}_{22}) = \left(\frac{1}{\beta} \left(\frac{\alpha}{2}\right)^2 \gamma, \frac{\alpha(1-\alpha)}{4}\gamma \right)$ and $(\hat{x}_{h1}, \hat{x}_{h2}) = \left(\frac{1}{2\beta} \frac{\alpha}{n}\gamma, \frac{1-\alpha\gamma}{n} \right)$, $\forall h = 3, ..., n+2$. For $\beta_1 = \beta_2 = \beta$, we verify that $(\check{e}_{11}, \check{e}_{21}) = (\hat{e}_{11}, \hat{e}_{21})$, $\check{p} = \hat{p}$ and $(\check{x}_{h1}, \check{x}_{h2}) = (\hat{x}_{h1}, \hat{x}_{h2})$, $\forall h$. *QED.* ■

Proposition 3 is analogous to proposition 1: if agents 1 and 2 possess the same productive technology, then acting as a leader or as a follower (for agent 1) does not change the overall market outcomes.

Proposition 7 *When $\beta_1 < \beta_2$, there is no Pareto domination between the Stackelberg-Cournot and the Cournot equilibria.*

Proof. Using (35)-(37) and the second part of the preceding proof (from which the utility levels reached at the Cournot equilibrium are deduced), we have $\hat{U}_1 - \check{U}_1 = \ln\left[\frac{(\beta_1+\beta_2)^2}{4\beta_1\beta_2}\right] > 0$ and $\hat{U}_2 - \check{U}_2 = 2\ln\left[1 + \frac{\beta_2(\beta_1-\beta_2)}{\beta_1^2}\right] < 0$ since $\beta_1 < \beta_2$. Moreover, we have $\hat{U}_h - \check{U}_h = \alpha \ln\left(\frac{\beta_1+\beta_2}{2\beta_1}\right) > 0$ since $\beta_1 + \beta_2 > 2\beta_1$, $h = 3, ..., n+2$. ■

Proposition 4 confirms proposition 2: the shift of agent 1 from a Cournot to a Stackelberg behavior is improving for her/him, deteriorating for her/his follower competitor (on the same side of the market) and improving for the other agents (on the other side of the market).

5 Welfare and merging about Stackelberg general equilibria

Confronting the SWE with the SCE, we can notably establish a Pareto domination (when $\gamma < 1$) and a merging (when γ tends to 1).

Proposition 8 *When $\gamma < 1$, the Stackelberg-Cournot equilibrium is Pareto dominated by the Stackelberg-Walras equilibrium.*

Proof. Consider Eq. (35)-(37). For both producers, we easily find $\hat{U}_h - \tilde{U}_h = \ln\gamma < 0$, $\forall h = 1, 2$, since $\gamma < 1$ for finite values of n and for $\alpha \in (0, 1)$. Additionally, $\hat{U}_h - \tilde{U}_h = \alpha\ln\gamma + (1-\alpha)\ln\left[\frac{1-\alpha\gamma}{1-\alpha}\right]$, $\forall h = 3, ..., n+2$. Then $sign\left(\hat{U}_h - \tilde{U}_h\right) = sign[\psi(\alpha)]$, where $\psi(\alpha) \equiv \alpha\ln(n-1) + (1-\alpha)\ln n - \ln(n-\alpha)$. This function is defined on $(0,1)$, with $\lim\limits_{\alpha\to 0}\psi(\alpha) \to 0$ and $\lim\limits_{\alpha\to 1}\psi(\alpha) \to 0$.

Moreover, $\frac{\partial \psi}{\partial \alpha} = \ln\left(\frac{n-1}{n}\right) + \frac{1}{n-\alpha}$, which satisfies $\frac{\partial \psi}{\partial \alpha} < 0$ (> 0) for $\alpha < \bar{\alpha}$ $(\alpha > \bar{\alpha})$, with $\bar{\alpha} = n - \frac{1}{\ln n - \ln(n-1)}$. Therefore $\psi(\alpha) < 0$, which leads to $\hat{U}_h - \tilde{U}_h < 0$, $\forall h = 3, ..., n+2$. *QED.* ■

This proposition shows that some agents may be better off as price takers and worse off as price makers. More generally, this configuration reveals the existence of a cooperation failure. The SWE Pareto dominates the SCE because all the strategic behaviors offset each other. The contractions of exchange and of production are general ($\tilde{y}_{11} + \tilde{y}_{21} > \hat{y}_{11} + \hat{y}_{21}$), but nobody succeeds in getting a better price (see Prop. 7).

Proposition 9 *When $\gamma \to 1$, the Stackelberg-Cournot equilibrium coincides with the Stackelberg-Walras equilibrium.*

Proof. Consider Eq. (32)-(34). We remark that (32)-(33) can be written $(\hat{x}_{11}, \hat{x}_{12}) = \gamma(\tilde{x}_{11}, \tilde{x}_{12})$ and $(\hat{x}_{21}, \hat{x}_{22}) = \gamma(\tilde{x}_{21}, \tilde{x}_{22})$. Then $\lim_{\gamma \to 1} (\hat{x}_{11}, \hat{x}_{12}) = (\tilde{x}_{11}, \tilde{x}_{12})$ and $\lim_{\gamma \to 1} (\hat{x}_{21}, \hat{x}_{22}) = (\tilde{x}_{21}, \tilde{x}_{22})$. And from (34), $\lim_{\gamma \to 1} \left(\frac{1}{2\beta_1} \frac{\alpha}{n} \gamma, \frac{1-\alpha\gamma}{n}\right) = \left(\frac{1}{2\beta_1} \frac{\alpha}{n}, \frac{1-\alpha}{n}\right) = (\tilde{x}_{h1}, \tilde{x}_{h2})$, $\forall h = 3, ..., n+2$. *QED.* ■

The merging between the SCE and the SWE is obtained when $\gamma = 1$, and two different circumstances can lead this parameter to take that value. The market power of agents endowed with good 2 nears to negligible when n tends to infinity, or when the good they own becomes intrinsically useless ($\alpha \to 1$). In both cases, the Cournotian behavior tends to the Walrasian one: using one's disappeared market power comes down to not using it[8].

It is meaningful to compare the preceding equilibria with the competitive equilibrium, when all agents behave as price takers. The market price, the production levels and the allocations for consumers $h = 3, ..., n+2$ are respectively, when $\beta_1 < \beta_2$ (see Appendix 2):

$$\left(\frac{p_1}{p_2}\right)^* = \beta_1. \tag{38}$$

$$(y_{11}^*, y_{21}^*) = \left(\frac{\alpha}{\beta_1}, 0\right), \ \forall h = 1, 2, \tag{39}$$

$$(x_{h1}^*, x_{h2}^*) = \left(\frac{\alpha}{\beta_1} \frac{1}{n}, \frac{1-\alpha}{n}\right), \ \forall h = 3, ..., n+2. \tag{40}$$

Proposition 10 *Consider y as the aggregate level of production for each equilibrium. When $\beta_1 \leq \beta_2$, $\check{y} < \breve{y} < y^*$ and $\hat{y} < \tilde{y} < y^*$. In particular, when $\beta_1 = \beta_2 = \beta$, $\hat{y} = \check{y} < \tilde{y} = \breve{y} < y^*$.*

Proof. From (39) we have $y^* = \frac{\alpha}{\beta_1}$. Consider first the other levels of production given in (12)-(13): $\tilde{y} = \tilde{y}_{11} + \tilde{y}_{21} = \frac{\alpha}{2\beta_1^2\beta_2}\left[\alpha\beta_2^2 + 2\alpha\beta_1^2 + (1-2\alpha)\beta_1\beta_2\right]$ and (28)-(29): $\hat{y} = \hat{y}_{11} + \hat{y}_{21} = \gamma\,\tilde{y}$. Moreover, from the proofs of Prop. (1) and (3), we

[8] The two results obtained comparing the SCE and the SWE correspond to analogous comparative results obtainable about the CE and the CWE.

know $\breve{y} = \breve{y}_{11} + \breve{y}_{21} = \frac{\alpha}{(\beta_1+\beta_2)^2}\left[\alpha(\beta_1^3+\beta_2^3)+\beta_1\beta_2^2+\beta_1^2\beta_2\right]$ and $\check{y} = \check{y}_{11} + \check{y}_{21} = \frac{\alpha\gamma}{(\beta_1+\beta_2)^2}\left[\beta_1+\beta_2+\alpha\left(\frac{\beta_1^2}{\beta_2}+\frac{\beta_2^2}{\beta_1}\right)\right]$, then $\check{y} = \gamma\breve{y}$. Firstly, we verify that y^* is the highest level of production when $\beta_1 \leq \beta_2$. When $\beta_1 < \beta_2$, $y^* > \tilde{y}$ if and only if $2\alpha\beta_1^2 + \alpha\beta_2^2 < (1+2\alpha)\beta_1\beta_2$, which is always verified for $\alpha \in (0,1)$. When $\beta_1 = \beta_2$, $y^* > \tilde{y}$ if and only if $\alpha < 1$. Additionally, when $\beta_1 < \beta_2$, $y^* > \breve{y}$ if and only if $\beta_1(\alpha\beta_1^2 - \beta_2^2) < (1-\alpha)\beta_2^3$, which is always verified for $\alpha \in (0,1)$. When $\beta_1 = \beta_2$, $y^* > \breve{y}$ if and only if $\alpha < 1$. Secondly, when $\beta_1 \leq \beta_2$, $\tilde{y} > \hat{y}$ and $\breve{y} > \check{y}$, with $\hat{y} = \gamma\ \tilde{y}$ and $\check{y} = \gamma\breve{y}$, since $\gamma < 1$. Thirdly, when $\beta_1 = \beta_2 = \beta$, we have $\tilde{y} = \breve{y} = \frac{\alpha(1+\alpha)}{2\beta}$ and $\hat{y} = \check{y} = \frac{\alpha(1+\alpha)}{2\beta}\gamma$. *QED.* ■

To put it in a nutshell: the more widespread the Walrasian behavior, the bigger the aggregate production. The competitive level or production is the highest one, and accordingly the competitive equilibrium price is the lowest one, as $p^* < \tilde{p} = \hat{p} < \breve{p} = \check{p}$ when $\beta_1 < \beta_2$, and $p^* < \tilde{p} = \hat{p} = \breve{p} = \check{p}$ when $\beta_1 = \beta_2$.

6 Conclusion

In the previous economy, trade is necessary to production and production is necessary to consumption for both oligopolists. Two types of asymmetries are here involved, in a general equilibrium framework under strategic interactions. The first one is an asymmetry within a sector, which captures the usual strategic interactions between an active leader and a follower. The second one is an asymmetry across sectors: agents of the first sector are producers, while the remaining agents of the other sector are consumers. But one salient feature is that agents endowed with no good will yet be able to exert market power through production activities. These two exchange and production asymmetries can lead to imperfectly competitive behaviors that create market distorsions.

The paper could be extended in the following directions. First, it could be interesting to increase the number of sectors in the economy, to generalize production activities in all sectors, and also to introduce non linearities in technologies. Second, optimal taxation policy could also be introduced in order to determine the conditions under which market distorsions caused by strategic interactions could be dampened.

A Appendix 1

We here follow an argument given by Gabszewicz and Michel (1997) for Cournot-Walras equilibria in pure exchange economies, in order to show that the strategy set of both oligopolists can be restricted. We here consider the follower, the same analysis prevailing for the leader. The program of the follower, called $(*)$, consists in solving $\underset{(x_{21},x_{22})}{Max}\ \alpha\ln(y_{21}-e_{21}+x_{21})+(1-\alpha)\ln x_{22}$ s.t. $p_1x_{21}+p_2x_{22} \leq \Pi_2(e_{21},y_{21})$ and $x_{21} \geq 0$ and $x_{22} \geq 0$. First, the positivity constraints on profits imply that $\Pi_2(e_{21},y_{21}) \geq 0$, which leads to $\frac{p_2}{p_1}\beta_2 y_{21} \leq e_{21}$.

Moreover, given y_{21}, any utility level that can be reached choosing $e_{21} \leq y_{21}$ can also be reached by determining e_{21} in such a way that the quantity $y_{21} - e_{21}$ kept for later consumption is at least equal to the competitive demand of good 2, that is $\alpha y_{21}\left(1-\beta_2\frac{p_2}{p_1}\right)$, in solving $\underset{(x_{21},x_{22})}{Max}\ \alpha \ln x_{21} + (1-\alpha)\ln x_{22}$ s.t. $p_1x_{21} + p_2x_{22} \leq p_1y_{21} - \beta_2p_2y_{21}$. Accordingly, we consider only strategies (e_{21}, y_{21}) satisfying the constraint $e_{21} \leq \alpha y_{21}\left(1+\beta_2\frac{p_2}{p_1}\right)$. Consider then the strategy set $\bar{E}_{21} = \left\{e_{21} \in \mathbb{R}^2_+ : \frac{p_2}{p_1}\beta_2 y_{21} \leq e_{21} \leq \alpha y_{21}\left(1+\beta_2\frac{p_2}{p_1}\right)\right\}$. If $e_{21} \leq \alpha y_{21}\left(1+\beta_2\frac{p_2}{p_1}\right)$, the solution to $(*)$ in E_{21} coincides with the solution to $(*)$ in $\bar{E}_{21}$, and this latter solution is given by $(x_{21}(p,e_{21}), x_{22}(p,e_{21})) = \left(0, \frac{p_1}{p_2}e_{21}\right)$. If $e_{21} \leq \alpha y_{21}\left(1+\beta_2\frac{p_2}{p_1}\right)$, the solution to the problem $(*)$ in E_{21} is given by $(x_{21}, x_{22}) = \left(e_{21} - \left[(1-\alpha) + \alpha\beta_2\frac{p_1}{p_2}\right], \frac{p_1}{p_2}\alpha y_{21}\left(1+\beta_2\frac{p_2}{p_1}\right)\right)$. The follower obtains a utility level equal to $U_2\left(y_{21} - e_{21} + x_{21}, \frac{p_1}{p_2}\alpha y_{21}\left(1+\beta_2\frac{p_2}{p_1}\right)\right) = U_2\left(\alpha y_{21}\left(1-\beta_2\frac{p_2}{p_1}\right), \frac{p_1}{p_2}\alpha y_{21}\left(1+\beta_2\frac{p_2}{p_1}\right)\right)$. Now consider that, if $e_{21} > \alpha y_{21}\left(1+\beta_2\frac{p_2}{p_1}\right)$, the strategy e_{21} is substituted by the strategy $e'_{21} = \alpha y_{21}\left(1+\beta_2\frac{p_2}{p_1}\right)$. Then $e'_{21} \in \bar{E}_{21}$ and, according to the fact that $(x_{21}, x_{22}) = \left(0, \frac{\Pi_2}{p_2}\right)$, the solution to the problem $(*)$ in the strategy set $\bar{E}_{21}$ is given by $(x_{21}(p,e'_{21}), x_{22}(p,e'_{21})) = \left(0, \frac{p_1}{p_2}\alpha y_{21}\left(1+\beta_2\frac{p_2}{p_1}\right)\right)$, so that the utility level of $(x_{21}(p,e'_{21}), x_{22}(p,e'_{21}))$ is given by $U_2\left(y_{21} - e'_{21}, \frac{p_1}{p_2}\alpha y_{21}\left(1+\beta_2\frac{p_2}{p_1}\right)\right) = U_2\left(\alpha y_{21}\left(1-\beta_2\frac{p_2}{p_1}\right), \frac{p_1}{p_2}\alpha y_{21}\left(1+\beta_2\frac{p_2}{p_1}\right)\right)$, which represents the level of utility reached by the follower at the optimal solution in E_{21}. Therefore, to determine the Stackelberg-Walras equilibrium, the strategy set of the follower may be restricted to $\bar{E}_{21}$. In this set, the optimal solution to problem $(*)$ is given by the solution to $\underset{(e_{21},y_{21})}{Max}\ V_2(e_{21}, y_{21}) = \alpha\ln(y_{21} - e_{21}) + (1-\alpha)\ln\left(\frac{p_1}{p_2}e_{21} - \beta_2 y_{21}\right)$.

B Appendix 2

We here determine the competitive equilibrium for the economy ξ. Each agent who owns good 2 solves the program $\underset{(x_{h1},x_{h1})}{Max}\ \alpha\ln x_{h1} + (1-\alpha)\ln x_{h2}$ s.t. $p_1x_{h1} + p_2x_{h2} \leq \frac{p_2}{n}$, $x_{21} \geq 0$ and $x_{22} \geq 0$. This leads to the competitive individual demands $(x_{h1}, x_{h2}) = \left(\frac{\alpha}{n}\frac{p_2}{p_1}, \frac{1-\alpha}{n}\right)$, $\forall h = 3, ..., n+2$, and then to the aggregate demand $(x_1, x_2) = \left(\alpha\frac{p_2}{p_1}, 1-\alpha\right)$. Each producer solves the program: $\underset{(y_{h1},x_{h1},x_{h2})}{Max}\ \alpha\ln x_{h1} + (1-\alpha)\ln x_{h2}$ s.t. $p_1x_{h1} + p_2x_{h2} \leq p_2y_{h1}\left(\frac{p_1}{p_2} - \beta_h\right)$, $x_{h1} \geq 0$, $x_{h2} \geq 0$ and $y_{h1} \geq 0$. If $\beta_1 \leq \beta_2$, then the supply correspondence of producer h, $h = 1, 2$, is given by: $y_{h1} = \left\{0 \text{ for } \frac{p_1}{p_2} < \beta_h,\ y_{h1} \in [0, \bar{y}_{h1}] \text{ for } \frac{p_1}{p_2} = \beta_h \text{ and } \bar{y}_{h1} \text{ for } \frac{p_1}{p_2} > \beta_h\right\}$, with $\bar{y}_{h1} = \max y_{h1}$. When $\beta_1 < \beta_2$, the market equilibrium price and the level of activity are determined by the aggregate demand function, where the supply of good 1 is perfectly elastic, i.e. for $\left(\frac{p_1}{p_2}\right)^* = \beta_1$ and $y_{11}^* = \alpha\frac{p_2}{p_1}$. The corresponding allocations are given by $(x_{h1}^*, x_{h2}^*) = (0,0)$, $\forall h = 1,2$, and $(x_{h1}^*, x_{h2}^*) = \left(\frac{\alpha}{\beta_1}\frac{1}{n}, \frac{1-\alpha}{n}\right)$, $\forall h = 3,...,n+2$. If $\beta_1 = \beta_2 = \beta$, then $\left(\frac{p_1}{p_2}\right)^* = \beta$ and $(y_{11}^*, y_{21}^*) = \left(\frac{1}{2}\frac{\alpha}{\beta}, \frac{1}{2}\frac{\alpha}{\beta}\right)$, with $(x_{h1}^*, x_{h2}^*) = (0,0)$, $\forall h = 1,2$ and $(x_{h1}^*, x_{h2}^*) = \left(\frac{\alpha}{\beta}\frac{1}{n}, \frac{1-\alpha}{n}\right)$, $\forall h = 3,...,n+2$.

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