On Shapley-Shubik Equilibria with Financial Markets[1]

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Abstract

In this paper we extend the Shapley-Shubik model to a two period financial economy and essentially address the question of the existence of an equilibrium. More precisely, we show the existence of nice equilibria, i.e. situation in which prices for both assets and commodities are strictly positive. Even if the general lines of the proof are largely influenced by the paper of Dubey-Shubik (1978), most of the arguments are new because of the financial nature of the economy. It forces us to deal with a generalized Nash equilibrium and to proscribe the use of arguments which only works with a single cash-in-advance constraint.

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Keywords :

strategic market game, financial market, Shapley-Shubik equilibria

J.E.L classification numbers : C72, D52, D66.

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1 Introduction

While there is, by now, an extensive literature devoted to games *à la* Shapley-Shubik (1977) associated with static economies, such inquiry was less often addressed in the case of intertemporal economies populated by finitely many players facing some exogenous uncertainty, and trading on financial markets, both in order to reallocate resources across periods and in order to insure themselves against uncertainty[1]. We therefore fill this gap by studying some aspects of a strategic game associated to a two-period financial economy. We are mainly concerned, in this paper, with the issues of existence but of course a number of other related issues stay on the research calendar[2].

The question of the existence is quite important in this case. In fact, even when security markets happen, by chance, to be complete, this result is new, and *cannot* be deduced from, say, Dubey & Shubik's (1978) paper. Indeed, it has been shown by Peck & Shell (1989) and latter by Weyers (2000), that the two market structures — a complete set of contingent commodities *versus* a complete set of Arrow securities — do *not* induce the same set of Nash equilibria. More specifically, Peck & Shell (1989) show that, when there is *fiat* money, an interior Nash-equilibrium allocation of the Arrow-securities game in which some income is transferred across states cannot be a Nash-equilibrium of the corresponding contigent-commodities game. Weyers (2000) completes this result by pointing out that the unique strategic equilibria which are common to the two market organizations are those involving no-trade in the first-period assets. In the case of commodity-money, like in this paper, which is closer to Shapley & Shubik's spirit, the non-equivalence is even more abrupt: an equilibrium of a game may not be an equilibrium of the other because it is simply not feasible !

Moreover, one can expect, following Koutsougeras (2003) or Koutsougeras &Papadopoulos (2004)[3], that arbitrage opportunities remain at equilibrium. This is not without consequences on our existence proof. The lack of no-arbitrage conditions first eliminates, unlike in a Walrasian GEI model, the ability to remove the financial assets by means of the Cass trick. It secondly bars the opportunity to eliminate redundant assets, i.e. we cannot assume that the return matrix is of full rank. Finally, and more severely, we know from the competitive setting that the absence of arbitrage opportunities contributes, in some sense, to the "compactification" of set of reachable portfolios. So under an unlimited short-sell assumption on the asset markets, one can expect that the strategy sets are not compact. This is why we introduce an assumption of secured lending[4] which states loosely speaking that the future debt induced by selling assets today must be covered in each states by the future holdings in *numéraire*.

[1] Strategic trading on financial markets has been explored, at our knowledge, by Peck & Shell (1989), Weyers (1999), Giraud & Stahn (2003), Giraud &Weyers (2004) and Koutsougeras & Papadopoulos (2004)

[2] Some of them are mentioned in the last section of the paper.

[3] In this last paper, Koutsougeras & Papadopoulos (2004) explicitly construct an example of an one commodity-S-state economy in which the no-arbitrage condition is not satisfied at equilibrium. They however never check the existence of an GEI Shapley-Shubik equilibrium. Our investigation therefore completes their work.

[4] See Sudderth, Karatzas & Shubik (1997)

To be more precise, we consider, in this paper, a two period economy with *numéraire* asset *à la* Geanokoplos-Polemarchakis (1986) and assume that both asset and commodity markets work *à la* Shapley-Shubik (1977).

The first period works like a standard market game. The different agents have the opportunity to bid on the different asset trading posts under a standard cash-in-advance constraint. But, in order to be consistent with the idea of borrowing and lending, we do not endow the agents with initial asset holdings and, from that point of view, allow *short sales.*

The second period works in a slightly different manner. Like in a standard GEI approach, we have to take into account the returns of the financial assets. But if we want to give the agents a real opportunity to modify their future trades, these returns must be paid before the commodity markets open. Since we work with *numéraire* assets, this modifies the cash-in-advance constraint of each player and give her the opportunity to really reallocate her consumption stream. But this modification of the Shapley-Shubik game is not without consequences. The second period state contingent cash-in-advance constraints can no more be viewed as simple restriction imposed on the strategy sets. These constraints are now affected by the asset allocation, or in other words, by the choice of the other players. A *Generalized Nash Equilibrium* in the sense of Debreu (1952) is therefore required.

But the introduction of assets also induces a new source of *bankruptcy.* An agent can go bankrupt either by biding too much in a given state or by having, in some state of nature, not enough cash to cover the debt that she contracted in the previous period. This is one of the reasons why we introduce a *secured lending* assumption. This last one states that the offers put on the different asset markets by a given agent when converted in future debt cannot be, state by state, greater than her initial holding in money. The reader also notices that this assumption introduces an upper bound on the short sales and thereafter gives us the opportunity to compactify the first period strategy sets without making use of a no arbitrage condition. But we must concede, following Peck & Shell (1989) (1990), that restrictions on short sales affect the equilibria in market games. It reduces, in particular, the degree of liquidity of the financial markets and contributes to inefficiency.

Within this setting, we basically concentrate our attention on the question of the existence of a *nice Shapley-Shubik equilibrium.* This proof relies, like the one of Dubey-Shubik (1978), on a standard fixed point argument contrary to the existence proof provided by Peck, Shell and Spear (1992) which makes use of a mod 2 degree approach.

This paper is organized in the following way. In the second section, we describe the basic setting of the paper. The third section is devoted to the definition of an equilibrium and its relation to our general equilibrium setting with financial markets. We show the existence of an $\varepsilon-$NE in section four and postpone the discussions the existence of 'interior nice' NEs to section five. Concluding comments and open questions are gathered in the last section.

2 The model

We first describe the economy underlying our framework, then we present the strategies available to each player and we conclude this presentation by the construction of the outcome function which specifies the mechanism on which this paper focuses.

2.1 The underlying economy

We consider a standard, two-period, exchange economy with *numéraire* assets (see Geanakoplos & Polemarchakis (1986) for details). At $t = 1$, one of $S \geq 1$ possible, uncertain states of Nature occurs. For the sake of simpler notations, $s = 0$ stands for $t = 0$.

There are $L + 1 \geq 1$ commodities at each state $s \geq 1$. In fact there are L standard consumption goods indexed by $\ell = 1, \ldots, L$ and an additional commodity which serves as a *numéraire*. For simplicity, we assume that spot commodity markets $\ell = 1, \ldots, L$ only open at $t = 1$, i.e. during the second market session[5], in each state of Nature $s \geq 1$. The commodity space is thus $\mathbb{R}_+^{S(L+1)+1}$and a consumption bundle $(x, m) \in \mathbb{R}_+^{S(L+1)+1}$ specifies an amount of commodities $x = (x_s)_{s=1}^{S} \in \mathbb{R}_+^{SL}$ as well as of *numéraires* $(m_s)_{s=0}^{S} \in \mathbb{R}_+^{S+1}$.

There are $N \geq 2$ household, indexed by $i = 1, \ldots, N$. Each of them is given a vector $\left(\omega^i, \mu^i\right) \in \mathbb{R}_{++}^{S(L+1)+1}$ of initial endowments, and each households chooses a commodity bundle in her consumption set $\mathbb{R}_+^{S(L+1)+1}$. The tastes of agent i are defined by an utility function $U^i : \mathbb{R}_{++}^{S(L+1)+1} \to \mathbb{R}$. Throughout this paper, we shall assume that each function U^i is continuous, quasi-concave, increasing, and satisfies the following standard boundary condition:[6] $\forall\, (x, m) \in \partial\mathbb{R}_+^{S(L+1)+1}$, $U^i(\omega_i, \mu^i) > U^i(x, m)$

The financial market is composed of $J \geq 1$ securities that can be exchanged at $t = 0$. One unit of asset j promises, at date $t = 1$, r_{sj} units of the *numéraire* commodity if state s occurs. The vector $r_s \in \mathbb{R}^J$ stands for $(r_{sj})_{j=1}^{J}$. A portfolio $\theta \in \mathbb{R}^J$ generates the future income transfers $R \cdot \theta$, where R is the $S \times J$ real matrix whose s^{th} row is r_s. Throughout the paper, we shall impose that $r_s \geq 0$ for any s, and, to avoid trivialities[7], that $\forall s\ \exists j\ :\ r_{sj} > 0$ i.e. in each state at least one asset pays positive returns and that $\forall j\ \exists s\ :\ r_{sj} > 0$ i.e. each asset pays positive returns in at least one state.

[5] We could allows for consumption of *non-numéraire* goods at $t = 0$, at the cost of more cumbsersome notations, and without any improvement of the economic content.

[6] This condition can in fact be weakened. A restriction which asserts, as in Dubey & Shubik (1978), that there are, for any good, at least to moneyed trader and at least two furnished traders would be enough. The boundary condition simplies says that the previous condition is met for all commodities and all agents and slighly simplifies the existence proof.

[7] The reader however notices that we do not assert that the return matrix R is of full rank. This directly follows from Koutsougas & Papadopoulos' results (2004). As the 'law of one price' is not satisfied, one can expect that an asset whose returns are a linear combination of the returns of some others may have a price which is not the same linear combination of the asset prices. In other words, redundant assets cannot be eliminated.

2.2 The strategies

Both the security markets of period 0 and the spot commodity markets of period 1 are organized according to the rules of a standard Shapley-Shubik game with *numéraire*. Each player's i action set is defined by: $A^i := \prod_{s=0}^{S} A^i_s$, where:

$$A^i_0 := \left\{ (b^i_j, q^i_j) \in (\mathbb{R}^J_+)^2 : \forall j = 1, \ldots, J, \ \sum_{j=1}^{J} b^i_j \leq \mu^i_0 \quad and \quad R \cdot \left(q^i_j\right)_{j=1}^{J} \leq \mu^i_1 \right\} \tag{1}$$

and

$$A^i_s := \left\{ (b^i_{\ell s}, q^i_{\ell s})_{\ell=1,\ldots,L} \in (\mathbb{R}^L_+)^2 : \forall \ell = 1, ..., L, \ q^i_{\ell s} \leq \omega^i_{\ell s} \right\}. \tag{2}$$

In words, q^i_j (resp. $q^i_{\ell s}$) is the quantity of asset j (resp. commodity ℓ) that trader i offers for sale in state 0 (resp. s). Similarly, b^i_j (resp. $b^i_{\ell s}$) is i's bid on security j (resp. commodity ℓ) in state 0 (resp. s). The constraint $\sum_{j=1}^{J} b^i_j \leq \mu^i_0$ represents the usual idea that individuals must finance their bids on securities by sales of *numéraire*. Notice, however, the dissymmetric treatment of the strategy sets at time $t = 0$ and $t = 1$. In fact:

1. While the offers in spot consumption goods are constrained by the physical impossibility to exceed one's initial endowments i.e. $q^i_{\ell s} \leq \omega^i_{\ell s}$, such a constraint cannot be imposed on assets. Indeed, consistently with the usual GEI model, we did not provide households with initial holdings in securities. A blind, physical constraint on assets would prevent any trade in securities!

2. While the bids in assets are constrained by the initial holding in *numéraire* in state 0 i.e. $\sum_{j=1}^{J} b^i_j \leq \mu^i_0$, a similar constraint for commodities cannot be imposed on the set of strategies. In fact, as in a usual GEI model, assets induce transfers of *numéraire* across time and states. This means, from a Shapley-Shubik point of view, that the agents have the ability to reorganize their future 'cash-in-advance' constraints. But the final asset holdings at the end of the first period result from strategic interactions. It is therefore impossible to restrict the bids for consumption goods independently of the strategies chosen by the other players.

3. Since an offer q^i_j of security j by trader i is easily interpreted as meaning that i is taking a short position with respect to security j and since q^i_j is not bounded by initial endowments, we have to make sure that too large short-sells are not allowed. In fact, in a standard GEI model, this job is done by the well-known no-arbitrage condition, But as we know from Koutsougeras & Papadopoulos (2004), such a trick can *not* hold in market games. This is why we assume that the offers made by an agent are bounded by her capacity of re-funding. Since these transfers of *numéraire* occur before the commodity markets open, we assume, in the spirit of "secured lending", that each short position must be covered by future initial holdings in *numéraire*, i.e. $R \cdot \left(q^i_j\right)_{j=1}^{J} \leq \mu^i_1$.

While point 2 can be easily managed by introducing a *Generalized Game* (see Debreu 1952), point 3 relies on a additional assumption which is not necessarily harmless. In fact , since Peck

& Shell (1989) we know that restrictions on short sales affect the equilibria in market games. It renders the markets less liquid (see Peck & Shell (1990)) in the sense that net trade can be small relative to gross trade and, hence relative to overall market volume, so that prices are almost unaffected by net trades. Moreover, it is obvious that this assumption forbids large short-sells especially if the returns are high and the future money holding low. From that point of view, it is not very surprising the no-arbitrage condition on the financial markets may not be satisfied due to this lack of liquidity or even that our NE does not converge to a GEI as the number of players grows because these restrictions are simply not present in these models. To our defense, we nevertheless argue that this is one of the most natural way to extend the notion of cash-in-advance to an intertemporal setting and from that point of view to stay as close as possible to a standard Shapley-Shubik game which a *numéraire* good[8].

2.3 Prices, endogenous constraints and outcomes

Given a profile of actions $a = (a^i, a^{-i}) \in A := A^i \times \prod_{j \in I \setminus \{i\}} A^j$, "price vectors" for assets and *non-numéraire* commodities are computed as follows:[9]

$$\begin{cases} \forall j = 1, \dots, J, \quad \pi_j = \dfrac{\sum_{i \in I} b^i_j}{\sum_{i \in I} q^i_j} := \dfrac{B_j}{Q_j} \\ \forall s = 1, \dots, S, \forall \ell = 1, \dots, L, \quad \pi_{\ell s} = \dfrac{\sum_{i \in I} b^i_{\ell s}}{\sum_{i \in I} q^l_{\ell s}} := \dfrac{B_{\ell s}}{Q_{\ell s}} \end{cases} \tag{3}$$

and if $Q_j = 0$ or $Q_{\ell s} = 0$, we set respectively $\pi_j = 0$ and $\pi_{\ell s} = 0$. The final holding of player i in security j is then given by:

$$\theta^i_j(a) := \begin{cases} \dfrac{b^i_j}{\pi_j} - q^i_j \text{ if } \pi_j \neq 0 \\ -q^i_j \text{ else} \end{cases}, \tag{4}$$

and his final allocation in *numéraire* is:

$$m^i_0(a) = \mu^i_0 - \sum_{j=1}^{J} b^i_j + \sum_{j=1}^{J} \pi_j q^i_j. \tag{5}$$

The trading process is very simple: the total amount $Q_j = \sum_{i \in I} q^i_j$ of security j which is offered is allocated to traders in proportion to their shares of the bids for asset j. Player's i share of the bids at trading-post j is $\frac{b^i_j}{B_j}$, where $B_j = \sum_{i \in I} b^i_j$. Thus, the *gross* receipts of security j for agent i are $\frac{b^i_j Q_j}{B_j}$, while the gross *numéraire* receipts on post j for player i is $q^i_j \cdot \pi_j$. Given our conventions, and consistently with the usual Shapley-Shubik games, it is assumed that, if there are no offers on post j, all *numéraire* bids are 'lost'. Finally, by a standard convention of

[8] We are nevertheless conscious that these drawbacks can be removed by shifting to a fiat money game *à la* Peck, Shell & Spear (1989). But this requires a quite different existence proof which is on our agenda.

[9] We define $\frac{x}{0} = 0$ for any real number x.

notation, we denote $B_j^{-i} := \sum_{h\in I\setminus\{i\}} b_j^h$ and $Q_j^{-i} := \sum_{h\in I\setminus\{i\}} q_j^h$ the total amount of bids and offers on market j except those of i.

At that point, we can now come back to point 2 of the preceding subsection. In fact, in each state $s \geq 1$, player i faces an additional 'cash-in-advance' constraint, meaning that she must finance her bids for commodities by her initial endowment in state *s-numéraire* and by the net income of her portfolio or, conversely, that her net holding in *numéraire*, in state s, i.e. after the transfers induced by the asset holdings, covers her bids i.e. $\forall s = 1, \dots, S$, $\sum_{\ell=1}^{L} b_{\ell s}^i \leq \mu_s^i + \sum_{j=1}^{J} r_{sj}\theta_j^i$. This means that the second period feasible bids are linked to the strategies chosen in the first period via the asset holding, we shall therefore consider a generalized game by introducing a correspondence $\alpha^i : A^{-i} \hookrightarrow A^i$ which describes the set of admissible strategies of agent i given the strategies of the other players. This correspondence is defined as follows $\forall i, \forall a^{-i} \in A^{-i}$,

$$\alpha^i(a^{-i}) := \left\{ a^i \in A^i : \sum_{\ell=1}^{L} b_{\ell s}^i \leq \mu_s^i + \sum_{j=1}^{J} r_{sj}\theta_j^i(a_i, a_{-i}) \text{ for all } s = 1, \dots, S \right\} \tag{6}$$

It finally remains to define the final allocation of player i in commodity $\ell = 1, \dots, L$ in state $s = 1, \dots, S$ given by:

$$x_{\ell s}^i(a) := \begin{cases} \omega_{\ell s}^i - q_{\ell s}^i + \dfrac{b_{\ell s}^i}{\pi_{\ell s}} & \text{if } \pi_{\ell s} \neq 0 \\ \omega_{\ell s}^i - q_{\ell s}^i & \text{else} \end{cases} \tag{7}$$

and her final allocation in *numéraire* good in state $s = 1, \dots, S$ given by

$$m_s^i(a) := \mu_s^i + \sum_{j=1}^{J} r_{sj}\theta_j^i(a) - \sum_{\ell=1}^{L} b_{\ell s}^i + \sum_{\ell=1}^{L} \pi_{\ell s} q_{\ell s}^i \tag{8}$$

3 The equilibrium : definition and discussion

If one denotes by $\varphi : A \to \left(\mathbb{R}_+^{LS} \times \mathbb{R}_+^{S+1}\right)^N$ the outcome function given by

$$\varphi(a) = \left(\varphi^i(a)\right)_{i\in I} = \left(\left(x_{\ell s}^i(a)\right)_{\ell=1,s=1}^{L,S}, \left(m_s^i(a)\right)_{s=0}^{S}\right)_{i\in I}$$

one typically deals with a generalized game $G = \langle \varphi(a), \left(\alpha^i(a^{-i})\right)_{i=1}^N \rangle$ and a natural extension of Shapley-Shubik approach to the case of financial market is to seek for a Nash Equilibrium (NE) :

Definition 1 *An action profile $\tilde{a} \in A$ is a NE of G iff the following holds for each $i \in I$:*

$$\forall a^i \in \alpha^i(\tilde{a}^{-i}), \qquad U^i\left(\phi^i(\tilde{a}^i, \tilde{a}^{-i})\right) \geq U^i\left(\phi^i(a^i, \tilde{a}^{-i})\right)$$

As usually in Shapley-Shubik's games, we observe that :

Remark 1 *The game is individually rational because each player has the opportunity to defend her initial endowments whatever the strategies of the others are by playing $a^i = 0$.*

Remark 2 *If everybody defend her initial endowments and decides nor to bid or to offer assets, then a trivial no-trade equilibrium occurs. In this case, as in standard Shapley-Shubik model, nobody can improve her situation by an unilateral deviation because she has no partner to trade with.*

From that point of view, we are sure that an equilibrium exists since the trivial no-trade satisfies this requirement. But within our two period setting with financial markets, we can even identify less trivial but nevertheless interesting cases of, say, '*extended' trivial equilibria.*

First of all, let us consider a economy in which L commodities are available tomorrow in each state and say that on the first L_1 markets are inactive in sense that they are nor bids or offers. Now take a Shapley-Shubik GEI equilibrium of an economy in which only $L - L_1$ commodity markets are available and in which the utility of the different players is the same as in the previous economy and in which the consumption levels for the L_1 first goods are identified to the initial endowments. It is now a matter of fact to observe that any equilibrium profile of strategies of the small economy completed by a zero bid and offer strategy of the first L_1 commodity markets is a NE of the initial economy. It makes therefore sense to restrict our attention to equilibria with the property that every commodity markets are active.

But this has also another consequence :

Remark 3 *The set of equilibria of our economy contains the set of Shapley-Shubik equilibria of the associated S-state one (numéraire) commodity economy. Moreover since our result is independent of the number of commodities, it also provides existence for this last case.*

From that point of view, our result both completes and extends the paper of Koutsougeras and Papadopoulos(2004) because our result makes sure the an equilibrium exists in their case and offers a way to extend their examples to economies with several commodities.

Let us now consider a polar situation in which each player decides nor to bid or to offer on the asset markets. If the commodity markets are nevertheless active, we can assert that :

Remark 4 *Let us assume the utility functions are of the VNM type and let us compute a Shapley-Shubik equilibrium of an one period economy in which each agent is characterized by her state contingent utility functions and owns ω_s^i. If $\bar{a}_s = \left(\overline{b^i_{\ell s}}, \overline{q^i_{\ell s}}\right)_{i=1,\ell=1}^{I,L}$ denotes this static equilibrium then any profile of strategies with the property that (i) $b^i_j = q^i_j = 0$ $\forall i, j$ and (ii) $\forall s$ $a_s = \bar{a}_s$, is a Shapley-Shubik equilibrium of our economy.*

We can therefore claim, at least in the VNM case, that the set of equilibrium is quite huge because it trivially contains all the equilibria of the associated state by state static economy. We even suspect that a similar result can be obtained without VNM utilities. In order to motivate that point, let us assume that nobody puts offers nor bids on the asset markets. By doing so, we deal, in some extend, with a standard one period Shapley-Shubik equilibrium. The only difference is that the players face several independent cash-in-advance constraints. But if one quickly goes through Dubey-Shubik 's existence proof (1978), one can be easily convinced that the existence result extends up to some slight adjustments. In fact, as long as the asset markets are inactive, the agents cannot transfer *numéraire* across states in order to change their cash-in-advance constraints. These constraints therefore remains independent and simply induce some additional restrictions on the strategy sets. This is, of course, not the case if at least one asset market is active.

In any case this previous discussion tells as that there is a large class of NE-equilibria in which several markets are inactive. By inactive, we mean, of course, a trading post on which at least one agent has the opportunity to put bids and/or offers, but cannot improve her situation by an unilateral deviation because the other players put neither a bid nor an offer. This is why we will focus in our existence proof on situations in which each market is active. In other words we say that :

Definition 2 *A NE is called "nice" iff all trading-posts are active.*

But let us remember that trivial NE occur when everybody bids 0 and supplies 0 on at least one trading-post. In other words, if all the players believe that a market will be extremely thin, then this market will be endogenously closed, and their beliefs will turn out to be self-justified. Consequently, the question is whether there exist non-trivial beliefs about market 'thickness' which are self-enforcing.

For this purpose, we will introduce, in the next section, an ε modification of G, denoted G_ε and prove the existence of an $\varepsilon - NE$. It finally remains in section 5 to check that the limit, as $\varepsilon \to 0$, of a sequence of NE's of G_ε is also a nice NE of G. This requires, since G is highly discontinuous, that the price sequence converges (at least for a subsequence) but not to 0.

4 The existence of $\varepsilon-$Nash Equilibria

In this section, we prove the existence of a $\varepsilon-$Nash equilibrium. But before entering into more technical arguments, let us quickly present the different steps of the proof and the major changes induced by the introduction of financial assets.

4.1 A guideline to the $\varepsilon-$Nash existence proof

First of all, let us introduce G_ε, the ε modification of G. This game is obtained by assuming as in a standard Shapley-Shubik game that, say, some outside agency places a fixed bid $\varepsilon > 0$ and a fixed supply $\varepsilon > 0$ in each of the $(J + LS)$ trading-posts. This rules out all the discontinuities in the outcome function and we write $\varphi_\varepsilon : A \to \left(\mathbb{R}_+^{LS} \times \mathbb{R}_+^{S+1}\right)^N$ for the corresponding strategic outcome functions. This one is given by :

$$\varphi_\varepsilon(a) = \left(\begin{array}{l} \left(\omega_{\ell s}^i - q_{\ell s}^i + \frac{(Q_{\ell s}+\varepsilon)}{(B_{\ell s}+\varepsilon)} b_{\ell s}^i\right)_{\ell=1,s=1}^{L,S} \\ \mu_0^i - \sum_{j=1}^J b_j^i + \sum_{j=1}^J \frac{(B_j+\varepsilon)}{(Q_j+\varepsilon)} q_j^i \\ \left(\mu_s^i + \sum_{j=1}^J r_{sj}\left(b_j^i \frac{q_j^i+Q_j^{-i}+\varepsilon}{b_j^i+B_j^{-i}+\varepsilon} - q_j^i\right) - \sum_{\ell=1}^L b_{\ell s}^i + \sum_{\ell=1}^L \frac{(B_{\ell s}+\varepsilon)}{(Q_{\ell s}+\varepsilon)} q_{\ell s}^i\right)_{s=1}^S \end{array} \right)_{i=1}^n \tag{9}$$

is obtained from (5), (7) and (8) by replacing B_j (resp. Q_j, $B_{\ell s}$, $Q_{\ell s}$) by $B_j+\varepsilon$ (resp. $Q_j+\varepsilon$, $B_{\ell s}+\varepsilon$, $Q_{\ell s}+\varepsilon$). The same can of course be done with (6) the correpondence which associates to the strategies of say player i her set of available strategies. This one becomes $\forall i, \forall a^{-i} \in A^{-i}$,

$$\alpha_\varepsilon^i(a^{-i}) := \left\{ a^i \in A^i : \sum_{\ell=1}^L b_{\ell s}^i \le \mu_s^i + \sum_{j=1}^J r_{sj} \underbrace{\left(b_j^i \frac{q_j^i+Q_j^{-i}+\varepsilon}{b_j^i+B_j^{-i}+\varepsilon} - q_j^i\right)}_{:=\theta_{j,\varepsilon}^i(a^i,a^{-i})} \text{ for all } s=1,\ldots,S \right\} \tag{10}$$

Thus :

Remark 5 *In order to get rid of the discontinuity induced by rule (7) and (8) and in order take into account that the second period feasible bids are link to the strategies chosen in the first period via the asset holding, we shall consider a generalized game G_ε and study the existence of a $\varepsilon-$Nash given by an action profile $\tilde{a} \in A$ such that for each $i \in I$:*

$$\forall a^i \in \alpha_\varepsilon^i(\tilde{a}^{-i}), \qquad U^i\left(\phi_\varepsilon^i(\tilde{a}^i, \tilde{a}^{-i})\right) \ge U^i\left(\phi_\varepsilon^i(a^i, \tilde{a}^{-i})\right)$$

Moreover it is a matter of fact to observe that if $\varepsilon \to 0$ and the different prices converges to non-zero prices then the limit profile of strategies is a nice NE of our economy.

Let us now observe that the introduction of a generalized game does not solve all the problem. At that point, we cannot assert that A_i is compact. This is were the "secured leading" assumption enters into the picture. In fact

Remark 6 *Under the "secured leading" assumption, i.e. $R \cdot \left(q_j^i\right)_{j=1}^J \le \mu_1^i$, we are sure that the offers for assets are bounded from above for every player. Since the bids on these markets are also bounded by the cash-in-advance constraint, we can expect that the bids of the commodity markets are also bounded whatever the strategies of the other players are. In other words, that we can restrict our attention to $C^i \subset A^i$ a compact convex subset of the set of strategies of each player*

We even show with standard arguments that $\alpha_\varepsilon^i(a^{-i})$ is continuous. But, if one takes as given the strategies chosen by the other players, one also notices that the restrictions on the feasible strategies of a given player are typically non-linear. Due to the transfers of *numéraire,* $\alpha_\varepsilon^i(a^{-i})$ is not necessarily convex-valued. This definitively rules out a direct study of the best response of each player subject to the constraints given by $\alpha_\varepsilon^i(a^{-i})$. To circumvent this problem, we simply move the study of the best reply from the strategy set to the feasible final allocation set by observing that :

Remark 7 *As long as (i) we consider an ε-equilibrium*[10] *and (ii) the bids and offers of the other players are taken as given, the final holdings in numéraire can be expressed as a function of the final consumption x^i, the portfolio holding θ^i and the strategy a^{-i} of the other players. More precisely $m^i(x,\theta,a^{-i})$ is given by :*

$$\begin{cases} m_0^i\left(\theta,a^{-i}\right)=\mu_0^i+\sum\limits_{j=1}^{J}\dfrac{\left(B_j^{-i}+\varepsilon\right)\theta_j}{Q_j^{-i}+\varepsilon-\theta_j} \\ \forall s=1,\ldots,S,\ m_s^i\left(x,\theta,a^{-i}\right)=\mu_s^i+\sum\limits_{\ell=1}^{L}\dfrac{\left(B_{\ell s}^{-i}+\varepsilon\right)\left(\omega_{\ell s}^i-x_{\ell s}\right)}{Q_{\ell s}^{-i}+\varepsilon+\omega_{\ell s}^i-x_{\ell s}}+\sum\limits_{j=1}^{J}r_{sj}\theta_j \end{cases}$$

The utility of agent i can therefore be written as $V^i(x,\theta,a^{-i}):=U^i(x,m^i\left(\theta,a^{-i}\right))$

Moreover, one also observes that :

Remark 8 *Since the set of reachable allocations $\left(x^i,\theta^i\right)$ is obtained by a correspondence B^i : $C^{-i}\hookrightarrow\mathbb{R}_+^{LS}\times\mathbb{R}^J$ defined on the strategies selected by the other players, i.e.*

$$B^i\left(a^{-i}\right)=\left\{\left(x^i,\theta^i\right)\in\mathbb{R}_+^{LS}\times\mathbb{R}^J:\exists a^i\in\alpha_\varepsilon^i(a^{-i}),\ x^i=x_\varepsilon^i\left(a^i,a^{-i}\right),\ \theta^i=\theta_\varepsilon^i\left(a^i,a^{-i}\right)\right\}$$

it remains to study $max_{(x,\theta)\in B^i(a^{-i})}V^i(x,\theta,a^{-i})$

If one checks that this new choice set is non-empty, compact, convex and continuous and that V^i is strictly quasi-concave on the allocation space, this new problem has a unique solution. It remains to move back to the strategy space in order to compute a best response correspondence and to observe that the basic assumption of Kakutani's fixed point theorem are satisfied. This proves the existence of a NE of the generalized game G_ε. Let us now detail the different step of the proof.

4.2 From non-convex strategy sets to convex choice sets

First of all let us make sure that we can restrict the strategy set of each agent to C^i, a non-empty, convex and compact set, and therefore only consider strategy profiles $\left(a^i,a^{-i}\right)\in C=C^i\times C^{-i}$ with $C^{-i}=\prod_{h\in I\setminus\{i\}}C^h$.

[10] *The fact that we considere a $\varepsilon-NE$ is crucial here. It makes sure that $\forall a^{-i}$, $Q_j^{-i}+\varepsilon-\theta_j=\frac{Q_j+\varepsilon}{B_j+\varepsilon}(B_j^{-i}+\varepsilon)>0$ and $Q_{\ell s}^{-i}+\varepsilon+\omega_{\ell s}^i-x_{\ell s}=\frac{Q_{\ell s}+\varepsilon}{B_{\ell s}+\varepsilon}(B_{\ell s}^{-i}+\varepsilon)>0$*

Lemma 1 *Under "secured lending", $\exists C^i \subset A^i$ a non-empty convex and compact set such that $\forall \varepsilon > 0$ and $\forall a^{-i} \in A^{-i}$, $\alpha^i_\varepsilon(a^{-i}) \subset C^i$.*

Proof : By *"secured lending",* agent i's asset offers verify $Rq \leq \mu^i_1$ and since $R \geq 0$ and R is non trivial (i.e. $\forall j\ \exists s,\ r_{sj} > 0$), we can say that $q^i_j \leq \frac{\max_{s=1,\ldots,S}\{\mu^i_s\}}{\min_{s=1,\ldots,S}\{r_{s,j}:r_{s,j}\neq 0\}}$. So, by taking the largest upper bound over i and j, we can say that $\forall i,j,\ q^i_j \leq \bar{q}$. Now, notice that the final holding $\theta^i_{\varepsilon,j}(B^{-i}_j, Q^{-i}_j, q^i_j, b^i_j) = \left(b^i_j\left(\frac{q^i_j+Q^{-i}_j+\varepsilon}{b^i_j+B^{-i}_j+\varepsilon}\right) - q^i_j\right)$ of player i in asset j decreases in q^i_j and B^{-i}_j while it increases in b^i_j and Q^{-i}_j. But q^i_j and B^{-i}_j are bounded from below by 0 while b^i_j and Q^{-i}_j are respectively bounded from above by μ^i_0 and $(n-1)\,\bar{q}$. It follows that $\theta^i_{\varepsilon,j}(B^{-i}_j, Q^{-i}_j, q^i_j, b^i_j) \leq \mu^i_0\left(\frac{(N-1)\bar{q}+\varepsilon}{\mu^i_0+\varepsilon}\right) \leq \theta^i_{\sup} := \mu^i_0 \max\left\{1, \frac{(N-1)\bar{q}}{\mu^i_0}\right\}$. Moreover we deduce from the state s strategic "cash-in-advance" constraint (5) and from the non negativity of the returns $r_{s,j}$ that each bid $b^i_{\ell s}$ in state s is bounded from above by $\bar{b}^i_s := \mu^i_s + \max_{j\in J}\{r_{j,s}\}\theta^i_{\sup} > 0$. It now remains to construct $C^i := A^i_0 \times \prod_{s=1}^S \bar{A}^i_s$ with :

$$\begin{cases} A^i_0 := \left\{(b^i_j, q^i_j) \in (\mathbb{R}^J_+)^2 : \forall j = 1,\ldots,J,\ \sum_{j=1}^J b^i_j \leq \mu^i_0 \quad \textit{and} \quad R\cdot\left(q^i_j\right)_{j=1}^J \leq \mu^i_1\right\} \\ \forall s = 1,\ldots,S \quad \bar{A}^i_s = \left\{(b^i_{\ell s}, q^i_{\ell s})_{\ell=1,\ldots,L} \in (\mathbb{R}^L_+)^2 : \forall \ell = 1,\ldots,L,\ q^i_{\ell s} \leq \omega^i_{\ell s} \text{ and } b^i_{\ell s} \leq \bar{b}^i_s\right\} \end{cases}$$

and to observe that $C^i \subset A^i$ is non-empty convex and compact with the property that $\forall\varepsilon > 0$ and $\forall a^{-i} \in A^{-i},\ \alpha^i_\varepsilon(a^{-i}) \subset C^i$.

■

Let us now restrict the correspondence $\alpha^i_\varepsilon(a^{-i})$ to C^{-i} and let us study its properties. Since $\alpha^i_\varepsilon(a^{-i})$ is defined through weak inequalities and since each agent always has the opportunity not to trade, we can assert that :

Lemma 2 $\forall i$, $\alpha^i_\varepsilon : C^{-i} \hookrightarrow C^i$ *is continuous and takes non-empty and compact values.*

Proof : Let us recall that :

$$\alpha^i_\varepsilon(a^{-i}) = \left\{ a \in \left(\mathbb{R}^{LS+J}_+\right)^2 : \begin{array}{l} q^i_{\ell s} \leq \omega^i_{\ell s},\ R\cdot\left(q^i_j\right)_{j=1}^J \leq \mu^i_1,\ \sum_{j=1}^J b^i_j \leq \mu^i_0 \\ \forall s,\ \sum_{\ell=1}^L b^i_{\ell s} \leq \mu^i_s + \sum_{j=1}^J r_{sj}\theta^i_{\varepsilon,j}\left(a^i, a^{-i}\right) \end{array} \right\}$$

It is immediate that $\alpha^i_\varepsilon(a^{-i})$ is non-empty because $0 \in \alpha^i_\varepsilon(a^{-i})$. Moreover as long as $\left(\omega^i, \mu^i\right) \gg 0$, this set also has a non-empty interior. Since $\alpha^i_\varepsilon(a^{-i})$ is defined through weak inequalities and θ^i_ε is continuous in a^i, $\alpha^i_\varepsilon(a^{-i})$ is closed-valued, hence compact because $\alpha^i_\varepsilon(a^{-i}) \subset C^i$ (see lemma 1) . But θ^i_ε is also continuous in a^{-i}, it is then a routine matter to verify that α^i_ε has a closed graph. Since α^i_ε maps into a compact set C^i, it follows (see Hildenbrand (1974 prop 2 p23) that α^i_ε is also upper-semi-continuous (u.s.c.). It now remains to verify that α^i_ε is lower-semi-continuous (l.s.c.). So let $\left(a^i_0, a^{-i}_0\right) \in C$ with the property that $a^i_0 \in \alpha^i_\varepsilon(a^{-i}_0)$ and let $\left(a^{-i}_n\right) \to a^{-i}_0$. Remember that $\theta^i_{\varepsilon,}$ is continuous in a^{-i}. As a consequence if $a^i_0 \in int\left(\alpha^i_\varepsilon(a^{-i}_0)\right)$, then $\exists N \in \mathbb{N}$, $\forall n > N$, $a^i_0 \in \alpha^i_\varepsilon(a^{-i}_n)$. In other words the sequence $a^i_n = 0$ for $n \leq N$ and $a^i_n = a^i_0$ for $n > N$ has the property that $\forall n$, $a^i_n \in \alpha^i_\varepsilon(a^{-i}_n)$ and $a^i_n \to a^i_0$. Let us now suppose that $a^i_0 \in \partial\left(\alpha^i_\varepsilon(a^{-i}_0)\right)$. Because $\alpha^i_\varepsilon(a^{-i})$ has a non-empty interior, let $a^i_1 \in int\left(\alpha^i_\varepsilon(a^{-i}_0)\right)$. As before $\exists N \in \mathbb{N}$, $\forall n > N$, $a^i_1 \in \alpha^i_\varepsilon(a^{-i}_n)$. Now consider the segment $\left[a^i_0, a^i_1\right]$, either it belongs to $\alpha^i_\varepsilon(a^{-i}_n)$ or it meets its boundary because $\left[a^i_0, a^i_1\right]$ is connected. We can therefore define $\left(a^i_n\right)$ by $a^i_n = 0$ for $n \leq N$ and for $n > N$, $a^i_n = a^i_0$ if $\left[a^i_0, a^i_1\right] \subset \alpha^i_\varepsilon(a^{-i}_n)$ or $a^i_n = \arg\min_{a^i\in\partial\alpha^i_\varepsilon(a^{-i}_n)\cap\left[a^i_0,a^i_1\right]} \left\|a^i - a^i_0\right\|$ else.

$\blacksquare$

But we cannot make sure that α^i_ε is a convex set. To circumvent this difficulty, we largely make use of remark 7. It order to precise this point, let us construct, $\forall i \in I$, the correspondence $B^i : C^{-i} \hookrightarrow \mathbb{R}^{LS}_+ \times \mathbb{R}^J$. which associates to any strategy of the other players the set of reachable commodity and asset allocations. This relation is given by :

$$B^i\left(a^{-i}\right) = \left\{\left(x^i,\theta^i\right) \in \mathbb{R}^{LS}_+ \times \mathbb{R}^J : \exists a^i \in \alpha^i_\varepsilon(a^{-i}),\, x^i = x^i_\varepsilon\left(a^i, a^{-i}\right),\, \theta^i = \theta^i_\varepsilon\left(a^i,a^{-i}\right)\right\}$$

We observe that :

Lemma 3 *$\forall i$, $B^i : C^{-i} \hookrightarrow \mathbb{R}^{LS}_+ \times \mathbb{R}^J$ is continuous and takes non-empty, compact, convex values.*

Proof : Let us first remember that $\forall a^{-i} \in C^{-i}$, $0 \in \alpha^i_\varepsilon(a^{-i})$ hence by the definition of $x^i_\varepsilon(a)$ and $\theta^i_\varepsilon(a)$ we know that $\left(\omega^i, 0\right) \in B^i\left(a^{-i}\right)$. Now let $K^i : C^{-i} \hookrightarrow A$ be such that $K^i(a^{-i}) := \left\{a \in A : a = \left(a^i, a^{-i}\right) \text{ for all } a^i \in \alpha^i_\varepsilon(a^{-i}) \text{ and } a^{-i} \in C^{-i}\right\}$. Since by lemma 2, $\alpha^i_\varepsilon(a^{-i})$ is continuous and takes compact values , it is obvious that $K^i(a^{-i})$ shares the same properties. Now remark that $B^i\left(a^{-i}\right) = \left\{\left(x^i,\theta^i\right) \in \mathbb{R}^{LS}_+ \times \mathbb{R}^J :\ x^i = x^i_\varepsilon\left(a\right),\, \theta^i = \theta^i_\varepsilon\left(a\right) \text{ for } a \in K^i(a^{-i})\right\}$. Because $x^i_\varepsilon(a)$ and $\theta^i_\varepsilon(a)$ are continuous functions, it follows that $B^i\left(a^{-i}\right)$ is continuous and takes compact values.

It remains to check that $\forall a^{-i} \in C^{-i}$, $B^i\left(a^{-i}\right)$ is a convex set. So let $\left(\bar{x}^i, \bar{\theta}^i\right)$ and $\left(\tilde{x}^i, \tilde{\theta}^i\right)$ be in $B^i\left(a^{-i}\right)$ and let $\lambda \in [0,1]$. If $\left(x^{i,c}, \theta^{i,c}\right) := \lambda\left(\bar{x}^i, \bar{\theta}^i\right) + (1-\lambda)\left(\tilde{x}^i, \tilde{\theta}^i\right) \in B^i\left(a^{-i}\right)$, the result is obtained. This means that we have to exhibit a strategy $a^i \in \alpha^i_\varepsilon(a^{-i})$ of player i such that $\left(x^{i,c}, \theta^{i,c}\right) = \left(x^i_\varepsilon\left(a^i, a^{-i}\right), \theta^i_\varepsilon\left(a^i, a^{-i}\right)\right)$. But before starting its construction, let us keep in mind that $\left(\bar{a}^i, a^{-i}\right)$ and $\left(\tilde{a}^i, a^{-i}\right)$ respectively induce $\left(\bar{x}^i, \bar{\theta}^i\right)$ and $\left(\tilde{x}^i, \tilde{\theta}^i\right)$ and let us observe that the allocation rule is, for each item of trade, decreasing in the offer and increasing and concave in the bid made by agent i.

Let us first concentrate on the commodity ℓs. We know by construction that $\bar{x}^i_{\varepsilon,\ell s} = x^i_{\varepsilon,\ell s}\left(\bar{b}^i_{\ell s}, \bar{q}^i_{\ell s}, a^{-i}\right)$ and $\tilde{x}^i_{\varepsilon,\ell s} = x^i_{\varepsilon,\ell s}\left(\tilde{b}^i_{\ell s}, \tilde{q}^i_{\ell s}, a^{-i}\right)$. So let us define $q^c_{\ell s} := \lambda \bar{q}^i_{\ell s} + (1-\lambda)\tilde{q}^i_{\ell s}$ and $q^-_{\ell s} := \min\left\{\bar{q}^i_{\ell s}, \tilde{q}^i_{\ell s}\right\}$ and let us observe that :

$$\omega^i_{\ell s} - \bar{q}^i_{\ell s} \leq \bar{x}^i_{\varepsilon,\ell s} \leq x^i_{\varepsilon,\ell s}\left(\bar{b}^i_{\ell s}, q^-_{\ell s}, a^{-i}\right) \text{ and } \omega^i_{\ell s} - \tilde{q}^i_{\ell s} \leq \tilde{x}^i_{\varepsilon,\ell s} \leq x^i_{\varepsilon,\ell s}\left(\tilde{b}^i_{\ell s}, q^-_{\ell s}, a^{-i}\right)$$

By a convex combination of this inequalities, we obtain that :

$$\omega^i_{\ell s} - q^c_{\ell s} \leq x^{i,c}_{\ell s} \leq \lambda x^i_{\varepsilon,\ell s}\left(\bar{b}^i_{\ell s}, q^-_{\ell s}, a^{-i}\right) + (1-\lambda) x^i_{\varepsilon,\ell s}\left(\tilde{b}^i_{\ell s}, q^-_{\ell s}, a^{-i}\right)$$

Since the allocation rule is concave in the bid of agent i, we deduce that :

$$\omega^i_{\ell s} - q^c_{\ell s} \leq x^{i,c}_{\ell s} \leq x^i_{\varepsilon,\ell s}\left(\lambda \tilde{b}^i_{\ell s} + (1-\lambda)\bar{b}^i_{\ell s}, q^-, a^{-i}\right)$$

So let $b^c_{\ell s} := \lambda \tilde{b}^i_{\ell s} + (1-\lambda)\bar{b}^i_{\ell s}$, and let be $f : [0, b^c_{\ell s}] \to \mathbb{R}$ be a continuous function given by $f(b) = x^i_{\varepsilon,\ell s}\left(b, q^-_{\ell s}, a^{-i}\right)$. Because $f(0) = \omega^i_{\ell s} - q^-_{\ell s}$ and $f(b^c_{\ell s}) \geq x^{i,c}_{\ell s}$, two cases happen :

- if $f(0) \leq x^{i,c}_{\ell s}$ there exists by the intermediate value theorem $b^i_{\ell s} \in [0, b^c_{\ell s}]$ such that $x^i_{\varepsilon,\ell s}\left(\left(b^i_{\ell s}, q^-_{\ell s}\right), a^{-i}\right) = x^{i,c}_{\ell s}$

- if $f(0) > x^{i,c}_{\ell s}$, choose $b^i_{\ell s} = 0$ and consider the function $g : [q^-_{\ell s}, q^c_{\ell s}] \to \mathbb{R}$ given by $g(q) = x^i_{\varepsilon,\ell s}(0, q, a^{-i})$. Because $g(q^-) = f(0) > x^{i,c}_{\ell s}$ and $g(q^c) = \omega^i_{\ell s} - q^c_{\ell s} \leq x^{i,c}_{\ell s}$, there exists again $q^i_{\ell s} \in [q^-_{\ell s}, q^c_{\ell s}]$ such that $x^i_{\varepsilon,\ell s}\left((0, q^i_{\ell s}), a^{-i}\right) = x^{i,c}_{\ell s}$

The reader now notices that this argument can be reproduced for each commodity in each state. The same argument even applies to the assets as long as one sets $\omega^i_{\ell s} = 0$ and replaces $x^i_{\varepsilon,\ell s}$ by $\theta^i_{\varepsilon,j}$. It follows that $\exists a_i \in \left(\mathbb{R}^{LS+J}\right)^2$ such that $\left(x^{i,c}, \theta^{i,c}\right) = \left(x^i_\varepsilon\left(a^i, a^{-i}\right), \theta^i_\varepsilon\left(a^i, a^{-i}\right)\right)$. It now simply remains to verify that $a^i \in \alpha^i_\varepsilon(a^{-i})$. To make that point let us first remember that, by construction, $\bar{a}^i$ and $\tilde{a}^i$ belong to $\alpha^i_\varepsilon(a^{-i})$. This means that :

- $R\left(\bar{q}^i_j\right)^J_{j=1} \leq \mu^i_1$ and $R\left(\tilde{q}^i_j\right)^J_{j=1} \leq \mu^i_1$. But we know that by construction $\left(q^i_j\right)^J_{j=1} \leq \left(q^c_j\right)^J_{j=1} := \lambda\left(\bar{q}^i_j\right)^J_{j=1} + (1-\lambda)\left(\tilde{q}^i_j\right)^J_{j=1}$ and that, $R \geq 0$. It follows that $R\left(q^i_j\right)^J_{j=1} \leq \mu^i_1$
- $\sum_{j=1}^J \bar{b}^i_j \leq \bar{m}^i_0$ and $\sum_{j=1}^J \tilde{b}^i_j \leq \bar{m}^i_0$ but, again, $b^i_j \leq b^c_j := \lambda\tilde{b}^i_j + (1-\lambda)\bar{b}^i_j$, hence $\sum_{j=1}^J b^i_j \leq \bar{m}^i_0$.
- $\forall \ell, s$, $\bar{q}^i_{\ell s} \leq \omega_{\ell s}$ and $\tilde{q}^i_{\ell s} \leq \omega_{\ell s}$. It immediately follows that $\forall \ell, s$, $q^i_{\ell s} \leq \lambda\bar{q}^i_{\ell s} + (1-\lambda)\tilde{q}^i_{\ell s} \leq \omega_{\ell s}$
- $\sum_{\ell=1}^L \bar{b}^i_{\ell s} \leq \bar{m}^i_0 + \sum_{j=1}^J r_{sj}\bar{\theta}^i_j$ and $\sum_{\ell=1}^L \tilde{b}^i_{\ell s} \leq \bar{m}^i_0 + \sum_{j=1}^J r_{sj}\tilde{\theta}^i_j$. Since by construction $\theta^i = \theta^i_\varepsilon\left(a^i, a^{-i}\right) = \lambda \cdot \bar{\theta}^i + (1-\lambda) \cdot \tilde{\theta}^i$ and $b^i_{\ell s} \leq b^c_{\ell s} := \lambda\tilde{b}^i_{\ell s} + (1-\lambda)\bar{b}^i_{\ell s}$, it follows that $\sum_{\ell=1}^L b^i_{\ell s} \leq \bar{m}^i_0 + \sum_{j=1}^J r_{sj}\theta^i_j$

Thus one can conclude that $a^i \in \alpha^i_\varepsilon(a^{-i})$ or, in other words, that $B^i\left(a^{-i}\right)$ is convex.

■

4.3 Back to strategies

Convexity can therefore be restored in the set of all reachable commodity and portfolio allocations as long as one takes as given, the strategies of the other players. We can now explicitly make use of remark 8 by observing that each agent optimal consumption and portfolio allocation in response of the strategies of the other players is given by :

$$OC^i\left(a^{-i}\right) = \left\{\left(\hat{x}^i, \hat{\theta}^i\right) \in \mathbb{R}^{LS}_+ \times \mathbb{R}^J : \left(\hat{x}^i, \hat{\theta}^i\right) \in \arg\max_{(x,\theta)\in B^i(a^{-i})} V^i(x, \theta, a^{-i})\right\}$$

Now remember that the *numéraire* is strictly desired in each state and that $U^i(x, m)$ is quasi-concave. Because $m^i(x, \theta, a^{-i})$ (see remark 7) is obviously strictly concave in (x, θ), one can expect that $V^i(x, \theta, a^{-i})$ is strictly quasi-concave with respect to (x, θ). But this implies that :

Lemma 4 *The optimal choice function $OC^i : C^{-i} \to \mathbb{R}^{LS}_+ \times \mathbb{R}^J$ is, for all i, a continuous function.*

Proof : Since $\hat{u}^i(x, \theta, a^{-i})$ is continuous and $B^i(a^{-i})$ is continuous and takes compact values, it is well known (see Hildenbrand (1974) p 30) that $OC^i : C^{-i} \hookrightarrow \mathbb{R}^{LS}_+ \times \mathbb{R}^J$ is non-empty, compact-valued and u.s.c. But $B^i(a^{-i})$ is also convex, it therefore remains to verify that $V^i(x, \theta, a^{-i})$ is strictly quasi-concave in (x, θ) in order to conclude that $OC^i(a^{-i})$ is a continuous function. So, let us choose $\lambda \in]0, 1[$, $\left(x', \theta'\right)$ and $(x", \theta")$ and and let us construct $(x^c, \theta^c) = \lambda\left(x', \theta'\right) + (1-\lambda)(x", \theta")$. Because $\frac{(B^{-i}_j+\varepsilon)\theta_j}{Q^{-i}_j+\varepsilon-\theta_j}$ and $\frac{(B^{-i}_{\ell s}+\varepsilon)(\omega^i_{\ell s}-x_{\ell s})}{Q^{-i}_{\ell s}+\varepsilon+\omega^i_{\ell s}-x_{\ell s}}$ are strictly concave function with respect respectively to θ_j and $x_{\ell s}$, it is a matter of fact to verify that $m(x,\theta,a^{-i})$ is strictly concave in (x,θ). Moreover because U^i is increasing in m and quasi-concave in (x,m), we have :

$$\begin{aligned} V^i(x^c,\theta^c,a^{-i}) &= U^i\left(x^c, m(x^c,\theta^c,a^{-i})\right) \\ &> U^i\left(x^c, \lambda m(x',\theta',a^{-i}) + (1-\lambda) m(x",\theta",a^{-i})\right) \\ &\geq \min\left\{U^i\left(x', m(x',\theta',a^{-i})\right), U^i\left(x", m(x",\theta",a^{-i})\right)\right\} \\ &= \min\left\{V^i\left(x',\theta',a^{-i}\right), V^i\left(x",\theta",a^{-i}\right)\right\} \end{aligned}$$

$V^i(x,\theta,a^{-i})$ is therefore strictly quasi-concave in (x,θ).

■

In order to construct the best reply of each player, it remains to associate to each optimal consumption and portfolio choice the set of all best replies, that is the set of all strategies whose the outcome is precisely this optimal choice. For that purpose, let us introduce $\forall a^{-i} \in C^{-i}$, the correspondence $\sigma^i := \left(\sigma^i_x \times \sigma^i_\theta\right) : \mathbb{R}^{LS}_+ \times \mathbb{R}^J \times C^{-i} \hookrightarrow C^i$ defined as follows :

$$\begin{aligned} \sigma^i_\theta(x,\theta,a^{-i}) \ &: \ = \left\{\left(b^i_j, q^i_j\right)^J_{j=1} \in \left(\mathbb{R}^J_+\right)^2 : \theta = \theta^i_\varepsilon\left(a^i, a^{-i}\right)\right\} \cap A^i_0 \\ \sigma^i_x(x,\theta,a^{-i}) \ &: \ = \left\{\left(b^i_{\ell s}, q^i_{\ell s}\right)^{L,S}_{\ell=1,s=1} \in \left(\mathbb{R}^{LS}_+\right)^2 : x = x^i_\varepsilon\left(a^{-i}, a^i\right)\right\} \cap A_1(\theta) \end{aligned}$$

with $A_1(\theta) = \left\{\left(b^i_{\ell s}, q^i_{\ell s}\right)^{L,S}_{\ell=1,s=1} \in \prod^S_{s=1} \bar{A}^i_s \mid \forall s = 1,\ldots,S \ \sum^L_{\ell=1} b^i_{\ell s} \leq \omega^i_{s0} + \sum^J_{j=1} r_{js}\theta^i_j\right\}$. Moreover we observe that :

Lemma 5 *For all $\left(x,\theta,a^{-i}\right)$ such that $(x,\theta) \in B^i(a^{-i})$, $\sigma^i : \mathbb{R}^{LS}_+ \times \mathbb{R}^J \times C^{-i} \hookrightarrow C^i$ takes non-empty, compact, convex values*

Proof : σ^i takes, by construction, non-empty values as long as $(x,\theta) \in B^i(a^{-i})$. Since (i) $\sigma^i(x,\theta,a^{-i})$ is a subset of C^i, a compact set, (ii) $\theta^i_\varepsilon\left(a^i,a^{-i}\right)$ and $x^i_\varepsilon\left(a^{-i},a^i\right)$ are both continuous functions and (iii) $A_1(\theta)$ is a closed set for each θ, we can also conclude that σ^i takes compact values. It remains to check that σ^i takes convex values. So let us first observe that the condition $\theta = \theta^i_\varepsilon\left(a^i,a^{-i}\right)$ is equivalent to $\forall j$, $q^i_j = \frac{Q^{-i}_j+\varepsilon-\theta_j}{B^{-i}_j+\varepsilon} b^i_j - \theta_j$. It follows that $\Theta(x,\theta,a^{-i}) := \left\{\left(b^i_j, q^i_j\right)^J_{j=1} \in \left(\mathbb{R}^J_+\right)^2 : \theta = \theta^i_\varepsilon\left(a^i,a^{-i}\right)\right\}$ is a convex set. By a similar argument, we can also assert that $\mathcal{X}(x,\theta,a^{-i}) := \left\{\left(b^i_{\ell s}, q^i_{\ell s}\right)^{L,S}_{\ell=1,s=1} \in \left(\mathbb{R}^{LS}_+\right)^2 : x = x^i_\varepsilon\left(a^{-i},a^i\right)\right\}$ is also convex. It now remains (i) to remember that A^i_0 is convex (ii) to notice that $A_1(\theta)$ is also convex. Since the intersection and the product of convex set remain convex, we conclude that σ^i takes convex values.

■

It remains to combine $OC^i\left(a^{-i}\right)$ and $\sigma\left(x,\theta,a^{-i}\right)$ in order to define each agent best reply $BR^i : C^{-i} \hookrightarrow C^i$ which is given by :

$$BR^i\left(a^{-i}\right) = \left\{a^i \in C^i : a^i \in \sigma^i\left(OC^i\left(a^{-i}\right), a^{-i}\right)\right\}$$

This correspondence is by lemma 5 non empty compact and convex valued. It simply remains to check that $BR^i\left(a^{-i}\right)$ is u.s.c. in order to make sure that $BR : C \to C$ with $BR(a) = \left(BR^i\left(a^{-i}\right)\right)_{i=1}^n$ admits a fixed point or, in other words to assert that :

Theorem 1 *Every generalized game G_ε admits an $\varepsilon - NE$.*

Proof : In order to verify that $BR^i\left(a^{-i}\right)$ is u.s.c., let us take a sequence $\left(a_n^{-i}\right) \to a_0^{-i}$ and a sequence $\left(a_n^i\right)$ with the property that $\forall n$, $a_n^i \in BR^i\left(a_n^{-i}\right)$. Because $a_n^i \in C^i$, a compact set, $\left(a_n^i\right)$ admits a converging subsequence and let a_0^i be its limit. Now let $\varphi_\varepsilon^i(a_n^{-i}, a_n^i)$ be the sequence of portfolios and commodity allocations obtained with (a_n^{-i}, a_n^i). By construction $\varphi_\varepsilon^i(a_n^{-i}, a_n^i) = OC^i\left(a_n^{-i}\right)$ for all n. But φ_ε^i and OC^i are both continuous function hence $\varphi_\varepsilon^i(a_0^{-i}, a_0^i) = OC^i\left(a_0^{-i}\right)$. It follows that $a_0^i \in BR^i\left(a_0^{-i}\right)$ or, in other words, that $BR^i\left(a^{-i}\right)$ is u.s.c.

■

5 The existence of "nice" Nash equilibria

The purpose of this section is to show that a "nice" NE can be obtained as a limit of a sequence of $\varepsilon - NE$ as $\varepsilon \to 0$. This must however be done carefully because a Shapley-Shubik game contains a lot of discontinuities which are not present in the sequence of $\varepsilon -$ *games* that we consider. Some preliminary remarks clarify this point by putting forward the idea that each $\varepsilon - NE$ price must be bounded from above and from below. This point is then checked for both the commodity and the asset prices. We finally prove our main existence result.

5.1 Some preliminary remarks

The existence of a nice Shapley-Shubik equilibrium follows, *a priori,* from a very simple idea : we take a sequence $\left(\left(\tilde{a}_\varepsilon^i\right)_{i=1}^I\right)_\varepsilon$ of $\varepsilon - NE$ and push $\varepsilon \to 0$. Lemma 1 even tells us that every $\varepsilon - NE$ belongs to C a compact subset of the strategy space whose definition is independent of ε. Hence :

Remark 9 *Every sequence $\left(\left(\tilde{a}_\varepsilon^i\right)_{i=1}^I\right)_\varepsilon$ of $\varepsilon - NE$ with the property that $\varepsilon \to 0$ admits a converging subsequence (css for short).*

But this does not mean that this subsequence converges to a NE of G, our Shapley-Shubik game with financial markets. In fact, G contains discontinuities in both the allocation rule and the player's correspondences of choice. So even if these ones are, by construction, removed in G_ε they reappear at the limit. This is why we really have to make sure that we deal, at the limit, with a Shapley-Shubik equilibrium.

In order to illustrate this point, let us take a converging sequence $a_\varepsilon \in A$ of actions with the property that $\forall i$, $a_\varepsilon^i \in \alpha_\varepsilon^i(a_\varepsilon^{-i})$, let us consider a generic trading post and let us denote respectively by $\varphi_{t,\varepsilon}^i, \varphi_{m,\varepsilon}^i$ player's i net trade and money allocation, and by p_ε the induced price sequence. Since the sequence of actions is, say, feasible, it is easy to check that both $\varphi_{t,\varepsilon}^i$ and

$\varphi^i_{m,\varepsilon}$ belong to a compact set, hence admit both a converging subsequence of limit $\varphi^i_{t,0}$ and $\varphi^i_{m,0}$. But is this $\varepsilon-$allocations at limit the same as the one predicted by the Shapley-Shubik rule ? The answer is unfortunately no. But :

Remark 10 *If the sequence p_ε of price on that generic trading post admits a css whose limit is different from 0, then $\varphi^i_{t,\varepsilon}$ and $\varphi^i_{m,\varepsilon}$ admit a css whose limit coincides with a Shapley-Shubik allocation. If this is not the case $\varphi^i_{t,0} \geq \varphi^i_t(a_0)$ and $\varphi^i_{m,0} \geq \varphi^i_i(a_0)$ since $\forall \varepsilon$, $\varphi^i_{t,\varepsilon} = b^i_\varepsilon \frac{Q_\varepsilon+\varepsilon}{B_e+\varepsilon} - q^i_\varepsilon \geq -q^i_\varepsilon$ and $\varphi^i_{m,\varepsilon} = q^i_\varepsilon \frac{B_e+\varepsilon}{Q_\varepsilon+\varepsilon} - b^i_\varepsilon \geq -b^i_\varepsilon$, some of these inequalities holding strictly for some sequences of actions*[11].

This drawback is common to all existence proofs derived from $\varepsilon-games$. But, since we work with a generalized Nash equilibrium, it may also happen that the choice set $\alpha^i_\varepsilon(a^{-i})$ obtained, say, at the limit does not coincide with $\alpha^i(a^{-i})$ the choice set computed at the limit with the Shapley-Shubik rule because the portfolio allocation matters. We face some, say, upper and lower semi-continuity problem. In order to clarify this point, let us observe :

Lemma 6 *Let $a^{-i}_\varepsilon \in A^{-i}$ be a converging sequence of actions of the other players whose limit is $a^{-i}_0 \in A^{-i}$*
(i) Let $a^i_\varepsilon \in \alpha^i_\varepsilon(a^{-i}_\varepsilon)$. If the induced sequence π^j_ε of asset prices admit a css whose limit is different from 0 then $\left(a^i_\varepsilon\right)$ admits a css whose limit verifies $a^i_0 \in \alpha^i(a^i_0)$.
(ii) If $a^i \in \alpha^i(a^{-i}_0)$ then there exists a sequence $\left(a^i_\varepsilon\right)$ with $a^i_\varepsilon \in \alpha^i_\varepsilon(a^{-i}_\varepsilon)$ and with the property that $a^i_\varepsilon \to a^i$ as $\varepsilon \to 0$.

Proof : In order to check (i), let us construct θ^i_ε the associated sequence of portfolio allocation. By lemma 1 (and its proof), we can say that both a^i_ε and θ^i_ε belongs to a compact set which is independently from ε. So if $(\pi_{j,\varepsilon})$ admits a css of limit $\pi_{j,0}$, we can select another subsequence with the property that $\theta^i_\varepsilon \to \theta^i_0$ and $a^i_\varepsilon \to a^i_0$. By remark 10 and since $\pi_{j,0} \neq 0$, we can even say that $\theta^i_0 = \theta^i(a^i_0, a^{-i}_0)$. Now remember $\forall \varepsilon$, $a^i_\varepsilon \in \alpha^i_\varepsilon(a^{-i}_\varepsilon)$ which is given by equation 10. By pushing at the limit, we obviously verify equation 6 so that $a^i_0 \in \alpha^i(a^{-i}_0)$.
Let us now move to (ii).If $a^i_0 \in int\left(\alpha^i(a^{-i}_0)\right)$, we know that :

$$\forall s = 1, \ldots, S,\ \sum_{\ell=1}^{L} b^i_{\ell s,0} < \mu^i_s + \sum_{j\in J^+} r_{sj}\left(b^{i,0}_j \frac{q^i_{j,0} + Q^{-i}_{j,0}}{b^i_{j,0} + B^{-i}_{j,0}} - q^i_{j,0}\right) + \sum_{j\in J^0} r_{sj}\left(-q^i_{j,0}\right)$$

with $J^0 = \left\{j = 1, \ldots, J : \pi_j(a^i_0, a^{-i}_0) = 0\right\}$ and J^+ it complement. Since $\forall j \in J^0$, $b^{i,0}_j \frac{q^i_{j,0}+Q^{-i}_{j,0}}{b^i_{j,0}+B^{-i}_{j,0}} \geq 0$ and the returns are non negative, we also have for any $\varepsilon > 0$

$$\forall s = 1, \ldots, S,\ \sum_{\ell=1}^{L} b^i_{\ell s,0} < \mu^i_s + \sum_{j\in J^+} r_{sj}\left(b^{i,0}_j \frac{q^i_{j,0} + Q^{-i}_{j,0}}{b^i_{j,0} + B^{-i}_{j,0}} - q^i_{j,0}\right) + \sum_{j\in J^0} r_{sj}\left(b^{i,0}_j \frac{q^i_{j,0} + Q^{-i}_{j,0} + \varepsilon}{b^i_{j,0} + B^{-i}_{j,0} + \varepsilon} - q^i_{j,0}\right)$$

[11] *Take for instance $b^\varepsilon_i = \varepsilon$, $B^\varepsilon_{-i} = 0$, $q^\varepsilon_i = \frac{\omega}{2}$ and $Q^\varepsilon_{-i} = 0$. One easely observes that $p_\varepsilon \to 0$ and that $\varphi^t_{i,\varepsilon} \to -\frac{\omega}{4} > \varphi^t_i(a^0) = -\frac{\omega}{2}$.*

Now observe by remark 10 that $\forall j \in J^+$ the sequence $\theta^i_{j,\varepsilon}(a^i_0, a^{-i}_\varepsilon) \to \theta^i_j(a^i_0, a^{-i}_0)$ and since $Q^{-i}_{j,\varepsilon} \to Q^{-i}_{j,0}$ and $B^{-i}_{j,\varepsilon} \to B^{-i}_{j,0}$ we can find a ε_0 small enough, such that

$$\forall s = 1, \ldots, S, \ \sum_{\ell=1}^{L} b^i_{\ell s,0} < \mu^i_s + \sum_{j=1}^{J} r_{sj}\theta^i_{j,\varepsilon}(a^i_\varepsilon, a^{-i}_\varepsilon)$$

i.e. such that $\forall \varepsilon < \varepsilon_0$, $a^i_0 \in \alpha^i_\varepsilon(a^{-i}_\varepsilon)$. In other words, the sequence $a^i_\varepsilon = 0$ for $\varepsilon \geq \varepsilon_0$ and $a^i_\varepsilon = a^i_0$ for $\varepsilon < \varepsilon_0$ has the property that $\forall \varepsilon$, $a^i_\varepsilon \in \alpha^i_\varepsilon(a^{-i}_\varepsilon)$ and $a^i_\varepsilon \to a^i_0$. Let us now suppose that $a^i_0 \in \partial\left(\alpha^i(a^{-i}_0)\right)$. In this case let us choose $a^i_1 \in int\left(\alpha^i(a^{-i}_0)\right)$. Since $\exists \varepsilon_1 > 0$, $\forall \varepsilon < \varepsilon_1$, $a^i_1 \in \alpha^i_\varepsilon(a^{-i}_\varepsilon)$, the segment $\left[a^i_0, a^i_1\right]$ for $\varepsilon < \varepsilon_1$ either belongs to $\alpha^i_\varepsilon(a^{-i}_\varepsilon)$ or meets its boundary. We can therefore define $\left(a^i_\varepsilon\right)$ by $a^i_\varepsilon = 0$ for $\varepsilon \geq \varepsilon_1$ and for $\varepsilon < \varepsilon_1$, $a^i_\varepsilon = a^i_0$ if $\left[a^i_0, a^i_1\right] \subset \alpha^i_\varepsilon(a^{-i}_\varepsilon)$ or $a^i_\varepsilon = \arg\min_{a^i \in \partial\alpha^i_\varepsilon(a^{-i}_\varepsilon) \cap \left[a^i_0, a^i_1\right]} \left\|a^i - a^i_0\right\|$ else.

■

At that point, it now becomes important to make sure that the $\varepsilon-$equilibrium sequence of prices both converges and reaches a non-zero limit otherwise we can neither make sure that the sequence of allocation converges to the Shapley-Shubik one, nor that the strategies obtained at the limit belongs to the correspondence of choice of the different players.

This is the motivation of the next two subsections. In fact we will show that each commodity and each asset price computed at an $\varepsilon - NE$ is both bounded from above and from below by a bound which is not only independent of ε but also of the choice of the $\varepsilon - NE$. These results are obtained by constructing appropriate deviation from a $\varepsilon - NE$ and by taking advantage from the fact that they cannot be improving. Two lemmata which are presented in the appendix help us to conclude.

5.2 Are the commodity prices bounded ?

We first begin with the commodity prices, say commodity ℓs, and show that $p^\varepsilon_{\ell s}$ admits a lower bound. The intuition is quite simple[12]. It starts from the observation that such a bound exists if nobody offers commodity ℓs. It therefore remains to consider a situation in which at least one agent makes a strictly positive offer and to construct a deviation in which she offers less.

Lemma 7 *There exists $c_{\ell s} > 0$ with the property that the $\varepsilon - NE$ price of each commodity satisfies $p^\varepsilon_{\ell s} \geq c_{\ell s}$*

Proof : Let us start with an $\varepsilon - NE$ profile of strategies $a = (a^i)$. If $Q_{\ell s} = 0$ it is obvious that $p^\varepsilon_{\ell s} = \frac{B_{\ell s}+\varepsilon}{\varepsilon} \geq 1$. So let us assume that $Q_{\ell s} > 0$. Two cases have to be considered :
Case 1 : $\exists i_0 \in I^+_q := \left\{i \in I : q^i_{\ell s} > 0\right\}$ for which $b^{i_0}_{\ell s} = 0$.
Suppose that agent i_0 decreases her offer $q^{i_0}_{\ell s}$ by $\Delta_1 \leq \min\left\{q^{i_0}_{\ell s}, \varepsilon, 1\right\}$. By doing so, she obtains less money and more commodities ℓs in state s. Since $\Delta_1 \leq \min\left\{q^{i_0}_{\ell s}, \varepsilon, 1\right\}$ and $b^{i_0}_{\ell s} = 0$, we observe that

[12] The reader however notices that the method developed in this paper is quite different from the one used by Dubey-Shubik (1978). Instead of increasing the bid of an agent we decreases her offer. By doing so we do not have to care about the budget constraint, a problem which is, in our case, much more complicated because the return of the assets enter into the story.

$(\Delta m_s) = -p^\varepsilon_{\ell s}\frac{Q^{-i_0}_{\ell s}+\varepsilon}{Q_{\ell s}+\varepsilon-\Delta_1}\Delta_1 \geq -p^\varepsilon_{\ell s}\Delta_1$ and $\left(\Delta x^{i_0}_{\ell s}\right) = \Delta_1$. Her allocation after deviation therefore verifies[13] $\left(x^{i_0}, m^{i_0}\right)_{after} \geq \left(x^{i_0}, m^{i_0}\right)_{before} + \Delta_1 \cdot (-p^\varepsilon_{\ell s} e_s + e_{\ell s})$. Since her utility is non-decreasing and her initial allocation was given at equilibrium, the right-hand side commodity bundle cannot be improving. By Lemma 10 and because $\Delta_1 \leq \min\left\{q^{iO}_{\ell s}, \varepsilon, 1\right\}$, we can say that either $\|-p^\varepsilon_{\ell s} e_s\| > \delta_1$ or $m^{i_0}_s - p^\varepsilon_{\ell s} < 0$. By Lemma 11, we also know that $m^{i_0}_s > K$, it follows that either $p^\varepsilon_{\ell s} > \delta_1$ or $p^\varepsilon_{\ell s} \geq K_1$.

<u>*Case 2*</u> : $\forall i \in I^+_q$, $b^i_{\ell s} > 0$

If $\#I^+_q \geq 2$, $\exists i_0 \in I^+_q$ for which $\frac{b^{i_0}_{\ell s}}{B_{\ell s}} \leq \frac{1}{2}$. So let us decreases $q^{i_0}_{\ell s}$ by $\Delta_2 \leq \min\left\{q^{i_0}_{\ell s}, \varepsilon, 2\right\}$. As in case 1, we have $\Delta m_s \geq -p^\varepsilon_{\ell s}\Delta_2$ and, concerning commodity ℓs, we observe that[14] $\Delta x_{\ell s} = \frac{B^{-i_0}_{\ell s}+\varepsilon}{B_{\ell s}+\varepsilon}\Delta_2 \geq \frac{1}{2}\Delta_2$. Thus $\Delta\left(x^{i_0}, m^{i_0}\right) \geq \frac{\Delta_2}{2} \cdot (-2p^\varepsilon_{\ell s} e_s + e_{\ell s})$ and we can conclude by the same argument as before that either $p^\varepsilon_{\ell s} > \frac{\delta_2}{2}$ or $p^\varepsilon_{\ell s} > \frac{K_2}{2}$. Finally if $\#I^+_q = 1$, let us again decreases the offer of $i_0 \in I^+_q$ by $\Delta_3 \leq \min\left\{q^{i_0}_{\ell s}, \varepsilon, 1\right\}$ .Since $Q_{\ell s} = q^{i_0}_{\ell s}$, we observe that $\Delta x^{i_0}_{\ell s} = \frac{B^{-i_0}_{\ell s}+\varepsilon}{B_{\ell s}+\varepsilon}\Delta_3 \geq \frac{\varepsilon}{B_{\ell s}+\varepsilon}\Delta_3$ and that[15] $\Delta m_s = -p^\varepsilon_{\ell s}\frac{\varepsilon}{q^{i_0}_{\ell s}+\varepsilon-\Delta_3}\Delta_3 \geq -p^\varepsilon_{\ell s}\frac{2\varepsilon}{q^{i_0}_{\ell s}+\varepsilon}\Delta_3$, hence $\Delta\left(x^{i_0}, m^{i_0}\right) \geq \frac{\varepsilon\Delta_3}{B_{\ell s}+\varepsilon} \cdot \left(-2\left(p^\varepsilon_{\ell s}\right)^2 e_s + e_{\ell s}\right)$ and again by both lemmata either $p^\varepsilon_{\ell s} > \sqrt{\frac{\delta_3}{2}}$ or $p^\varepsilon_{\ell s} > \sqrt{\frac{K_3}{2}}$

We can therefore conclude that $p^\varepsilon_{\ell s} \geq c_{\ell s} := \min\left\{1, \delta_1, K_1, \frac{\delta_2}{2}, \frac{K_2}{2}, \sqrt{\frac{\delta_3}{2}}, \sqrt{\frac{K_3}{2}}\right\}$

■

Let us now seek for an upper bound. The method is here closer to the one developed by Dubey-Shubik (1978) and can be summarized as follows. We select an agent who offers less than a half of the aggregate offer. If she has offered more then a half of her endowments, the price $p^\varepsilon_{\ell s}$ is automatically bounded from above. If this is not the case, we construct a deviation in which she increases her offer and obtain the required upper bound. Thus :

Lemma 8 *There exists $C_{\ell s} > 0$ with the property that the $\varepsilon - NE$ price of each commodity satisfies $p^\varepsilon_{\ell s} \leq C_{\ell s}$*

Proof : Let us consider an $\varepsilon - NE$ profile of strategies $a = (a^i)$ and let us choose an agent i_0 for which $\frac{q^{i_0}_{\ell s}}{Q_{\ell s}} \leq \frac{1}{2}$. If $q^{i_0}_{\ell s} > \frac{\omega^i_{\ell s}}{2}$ we know (see the proof of lemma 1) that $\forall i \in I$, $b^i_{\ell s} \leq \bar{b}^i_s$, so that $B_{\ell s} \leq \bar{B}_{\ell s}$. It follows that $p^\varepsilon_{\ell s} = \frac{B_{\ell s}+\varepsilon}{Q_{\ell s}+\varepsilon} \leq \frac{\bar{B}_{\ell s}+\varepsilon}{\frac{\omega^i_{\ell s}}{2}+\varepsilon} \leq \max\left\{1, 2\frac{\bar{B}_{\ell s}}{\omega^i_{\ell s}}\right\}$. If $q^{i_0}_{\ell s} \leq \frac{\omega^i_{\ell s}}{2}$, let us increase $q^{i_0}_{\ell s}$ by $\Delta_4 = \min\left\{\varepsilon, \frac{\omega^i_{\ell s}}{2}, 2\left(p^\varepsilon_{\ell s}\right)^{-1}\right\}$. We obtain $\left(\Delta x^{i_0}_{\ell s}\right) = -\frac{B^{-i_0}_{\ell s}+\varepsilon}{B_{\ell s}+\varepsilon}\Delta_4 \geq -\Delta_4$ and $\left(\Delta m^{i_0}_s\right) = p^\varepsilon_{\ell s}\frac{Q^{-i_0}_{\ell s}+\varepsilon}{Q_{\ell s}+\varepsilon+\Delta_4}\Delta_4$. Since $\frac{q^{i_0}_{\ell s}}{Q_{\ell s}} \leq \frac{1}{2}$ and $\Delta_4 \leq \varepsilon$, we even have[16] $\left(\Delta m^{i_0}_s\right) \geq \frac{1}{2}p^\varepsilon_{\ell s}\Delta_4$. Hence $\Delta\left(x^{i_0}, m^{i_0}\right) \geq \frac{p^\varepsilon_{\ell s}\Delta_4}{2} \cdot (-\frac{2}{p^\varepsilon_{\ell s}} e_{\ell s} + e_s)$ and by lemmata 10 and 11, we can assert that either $p^\varepsilon_{\ell s} < \frac{2}{\delta_4}$ or $p^\varepsilon_{\ell s} < \frac{2}{K_4}$. Thus $p^\varepsilon_{\ell s} \leq C_{\ell s} := \max\left\{1, 2\frac{\bar{B}_{\ell s}}{\omega^i_{\ell s}}, \frac{2}{\delta_4}, \frac{2}{K_4}\right\}$

■

[13] We denote by $e_{\ell s}$ a vector whose components are all 0 except for commodity ℓs where the component is set to 1. The vector e_s is defined in the same way but it concerns the holding in money in state s.

[14] Since $\frac{b^{i_0}_{\ell s}}{B_{\ell s}} \leq \frac{1}{2}$ we can say that $B^{-i_0}_{\ell s} \geq \frac{B_{\ell s}}{2}$. It follows that $\frac{B^{-i_0}_{\ell s}+\varepsilon}{B_{\ell s}+\varepsilon}\Delta_2 \geq \frac{\frac{B_{\ell s}}{2}+\varepsilon}{B_{\ell s}+\varepsilon}\Delta_2 \geq \frac{\frac{1}{2}(B_{\ell s}+2\varepsilon)}{B_{\ell s}+2\varepsilon}\Delta_2 = \frac{1}{2}\Delta_2$.

[15] Let us first observe that $\Delta m_s \geq -p^\varepsilon_{\ell s}\frac{2\varepsilon}{q^{i_0}_{\ell s}+2\varepsilon-\Delta_3}\Delta_3$ because $\frac{-\varepsilon}{q^{i_0}_{\ell s}+\varepsilon}$ decreases with ε. Moreover since $\Delta_3 \leq \varepsilon$, it follows $\Delta m_s \geq -p^\varepsilon_{\ell s}\frac{2\varepsilon}{q^{i_0}_{\ell s}+\varepsilon}\Delta_3$

[16] Observe that $\frac{q^{i_0}_{\ell s}}{Q_{\ell s}} \leq \frac{1}{2}$ implies that $\frac{Q^{-i_0}_{\ell s}}{Q_{\ell s}} \geq \frac{1}{2}$. It follows that $\Delta m^{i_0}_s \geq p^\gamma_{\ell s}\frac{\frac{1}{2}Q_{\ell s}+\varepsilon}{Q_{\ell s}+\varepsilon+\Delta_4}\Delta_4$. Now remember that $\Delta_4 \leq \varepsilon$. We can therefore say that $\Delta m^{i_0}_s \geq p^\varepsilon_{\ell s}\frac{\frac{1}{2}Q_{\ell s}+\varepsilon}{Q_{\ell s}+2\varepsilon}\Delta_4 = \frac{1}{2}p^\varepsilon_{\ell s}\Delta_4$.

5.3 The asset prices are also bounded

Let us first study the existence of lower bound for the asset prices. By chance, we can obtain this bound by applying the same method as in lemma 7. We however has to take care to the asset structure since any decrease of an offers for assets leads to a transfer of money from the present to the future.

Corollary 1 *There exists $k_j > 0$ with the property that the $\varepsilon - NE$ price of each asset satisfies $\pi_j^\varepsilon \geq k_j$*

Proof : Let us keep in mind the proof of lemma 7 and let us introduce a decrease of the agent i_0's offer in asset j. Hence any change in $\left(\Delta x_{\ell s}^{i_0}\right)$ can now be viewed as a change $\left(\Delta\theta_j^{i_0}\right)$ of her holding in asset j. Since $\theta_j^{i_0}$ increases and $\forall s, j\ r_{s,j} \geq 0$, she does not go bankrupt after deviation. So let us denote by (Δm_1) the change in her final holding in money in each future state. Since the asset structure is non-trivial (i.e $\forall j,\ \exists s,\ r_{s,j} > 0$), $\exists s(j) := \arg\min_{s=1,\ldots,S}\{r_{sj} : r_{sj} > 0\}$, we can say that $(\Delta m_1) \geq r_{s(j),j}\left(\Delta\theta_j^{i_0}\right) \cdot e_{s(j)}$. So by making the right substitution, we can conclude that $\pi_j^\varepsilon \geq k_j = r_{s(j),j}\min\left\{\frac{1}{r_{s(j),j}}, \delta_1', K_1', \frac{\delta_2'}{2}, \frac{K_2'}{2}, \sqrt{\frac{\delta_3'}{2}}, \sqrt{\frac{K_3'}{2}}\right\}$

■

A similar argument does not work for the construction of an upper bound since the proof lemma 8 heavily relies on the existence of initial endowments and this is not the case for financial assets. An other method is therefore required.

So let us observe that the price of, say, asset j is bounded from above if all bids are zero. We can therefore restrict our attention to a situation in which at least one agent posts a bid and we can look at the consequences of a decrease of this quantity. By doing so she initiates a transfer of money from the future to the present, but she also takes the risk of running into bankruptcy. This is why we compute a more complicated deviation in which her bids on the future commodity market also adjust in order to prevent this bad even. This is made possible by the secured lending assumption which guaranties that any player who makes a bid on an asset trading post always owns some money before trading commodities. This is why if she initially spends all her money in a given state, she is always able to decrease one of her bids. But, by doing so, we obtain an upper bound which depends on the commodity prices which are known to be bounded. We can therefore say that :

Lemma 9 *There exists $K_j > 0$ with the property that the $\varepsilon - NE$ price of each commodity satisfies $\pi_j^\varepsilon \leq K_j$*

Proof : If $B_j = 0$ the result is obvious since $\pi_j^\varepsilon = \frac{\varepsilon}{Q_j+\varepsilon} \leq 1$. So let us assume that $B_j > 0$. Two cases must be considerate :

<u>*Case 1*</u> *:* $\exists i_0 \in I_b^+ := \left\{i \in I : b_j^i > 0\right\},\ q_j^{i_0} = 0$

Let us assume that agent i_0 decreases her bids $b_j^{i_0}$ by $\Delta_5 \leq \min\left\{b_j^{i_0}, \varepsilon\right\}$. Since $q_j^{i_0} = 0$, we observe that $(\Delta m_0) = \Delta_5$ and $\left(\Delta\theta_j^{i_0}\right) = -\frac{1}{\pi_j^\varepsilon}\frac{B_j^{-i_0}+\varepsilon}{B_j+\varepsilon-\Delta_5}\Delta_5 \geq -\frac{1}{\pi_j^\varepsilon}\Delta_5$. Now define $S_+ = \{s \in S : r_{sj} > 0\}$ and construct

$S_+^b := \left\{ s \in S_+ : \sum_{\ell=1}^L b_{\ell s}^{i_0} = \mu_s^{i_0} + \sum_{j=1}^J r_{sj}\theta_j^{i_0} \right\}$ and $S_+^u := S_+ \backslash S_+^b$ the sets of states in which the cash in advance constraint is respectively binding and unbinding. In the last case, there is no bankruptcy problem as long as Δ_5 is smaller than $\min_{s\in S_+^u} \left\{ \frac{\pi_j^\varepsilon}{r_{sj}} \left(\sum_{\ell=1}^L b_{\ell s}^{i_0} - \mu_s^{i_0} - \sum_{j=1}^J r_{sj}\theta_j^{i_0} \right) \right\}$, hence $\forall s \in S_+^u$, $(\Delta m_s) \geq -\frac{r_{sj}}{\pi_j^\varepsilon}\Delta_5$. But for $s \in S_+^b$, bankruptcy occurs after this deviation. But the secured lending assumption implies that $\mu_s^{i_0} + \sum_{j=1}^J r_{sj}\theta_j^{i_0} > 0$, so that $\forall s \in S_+^b$, $\exists \ell(s)$, $b_{\ell(s),s}^{i_0} > 0$. By choosing Δ_5 smaller than $\min_{s\in S_+^b} \left\{ b_{\ell(s),s}^{i_0} \right\} \times \min \left\{ \frac{\pi_j^\varepsilon}{\max_{s\in S_+^b}\{r_{sj}\}}, 1 \right\}$, we can decrease $b_{\ell(s),s}^{i_0}$ by $\left(\Delta b_{\ell(s),s}^{i_0} \right) = \frac{r_{sj}}{\pi_j^\varepsilon}\Delta_5$ in order to prevent bankruptcy. As a consequence $\forall s \in S_+^b$, $(\Delta m_s) = \frac{Q_{\ell(s),s}^{-i_0}+\varepsilon}{Q_{\ell(s),s}+\varepsilon}\frac{r_{sj}}{\pi_j^\varepsilon}\Delta_5 \geq 0$ and $\left(\Delta x_{\ell(s),s} \right) = -\frac{1}{p_{\ell(s),s}^\varepsilon}\frac{B_{\ell(s),s}^{-i_0}+\varepsilon}{B_{\ell(s),s}+\varepsilon-\left(\Delta b_{\ell(s),s}\right)}\left(\Delta b_{\ell(s),s} \right) \geq -\frac{1}{p_{\ell(s),s}^\varepsilon}\frac{r_{sj}}{\pi_j^\varepsilon}\Delta_5$. To sum up, we have :

$$\Delta\left(x^{i_0}, m^{i_0}\right) \geq \Delta_5 \cdot \left(-\left(\sum_{s\in S_+^u} \frac{r_{sj}}{\pi_j^\varepsilon} e_s + \sum_{s\in S_+^b} \frac{1}{p_{\ell(s),s}^\varepsilon}\frac{r_{sj}}{\pi_j^\varepsilon} e_{\ell(s),s} \right) + e_0 \right)$$

By Lemmata 10, 11 and 7, we know that either :

- $\frac{1}{\pi_j^\varepsilon}\sqrt{\sum_{s\in S_+^u} r_{sj}^2 + \sum_{s\in S_+^b}\left(\frac{r_{sj}}{p_{\ell(s),s}^\varepsilon} \right)^2} > \delta_5 \Rightarrow \pi_j^\varepsilon < \frac{1}{\delta_5}\left(\sum_{s\in S_+^u} r_{sj}^2 + \sum_{s\in S_+^b}\left(\frac{r_{sj}}{c_{\ell(s),s}} \right)^2 \right)^{\frac{1}{2}}$.
- $\exists s \in S_+^u$, $m_s^{i_0} - \frac{r_{sj}}{\pi_j^\varepsilon} < 0 \Rightarrow \frac{r_{sj}}{\pi_j^\varepsilon} > K_5 \Rightarrow \pi_j^\varepsilon < \frac{r_{sj}}{K_5}$
- $\exists s \in S_+^b$, $x^{i_0} - \frac{1}{p_{\ell(s),s}^\varepsilon}\frac{r_{sj}}{\pi_j^\varepsilon} < 0 \Rightarrow \pi_j^\varepsilon < \frac{1}{c_{\ell(s),s}}\frac{r_{sj}}{K_5}$

So let us define $K_j^1 := \max \left\{ \frac{1}{\delta_5}\left(\sum_{s\in S_+^u} r_{sj}^2 + \sum_{s\in S_+^b}\left(\frac{r_{sj}}{c_{\ell(s),s}} \right)^2 \right)^{\frac{1}{2}}, \left(\frac{r_{sj}}{K_5} \right)_{s\in S_+^u}, \left(\frac{1}{c_{\ell(s),s}}\frac{r_{sj}}{K_5} \right)_{s\in S_+^u} \right\}$

<u>*Case 2*</u> : $\forall i \in I_b^+ := \left\{ i \in I : b_j^i > 0 \right\}$, $q_j^i > 0$

If $\#I_b^+ \geq 2$, $\exists i_0 \in I_b^+$ with the property that $\frac{q_j^{i_0}}{Q_j} \leq \frac{1}{2}$. So let us decrease her bid $b_j^{i_0}$ by $\Delta_6 \leq \min\left\{ b_j^{i_0}, \varepsilon \right\}$. It follows that[17] $\Delta m_0 = \frac{Q_j^{-i_0}+\varepsilon}{Q_j+\varepsilon}\Delta_6 \geq \frac{1}{2}\Delta_6$ and $\Delta\theta_j^{i_0} \geq -\frac{1}{\pi_j^\varepsilon}\Delta_6$. By reproducing a similar argument as in case 1 in order to escape bankruptcy, we finally obtain :

$$\Delta\left(x^{i_0}, m^{i_0}\right) \geq \frac{\Delta_6}{2} \cdot \left(-2\left(\sum_{s\in S_+^u} \frac{r_{sj}}{\pi_j^\varepsilon} e_s + \sum_{s\in S_+^b} \frac{1}{p_{\ell(s),s}^\varepsilon}\frac{r_{sj}}{\pi_j^\varepsilon} e_{\ell(s),s} \right) + e_0 \right)$$

and construct a new bound $K_j^2 = \max \left\{ \frac{2}{\delta_6}\left(\sum_{s\in S_+^u} r_{sj}^2 + \sum_{s\in S_+^b}\left(\frac{r_{sj}}{c_{\ell(s),s}} \right)^2 \right)^{\frac{1}{2}}, \left(\frac{2r_{sj}}{K_6} \right)_{s\in S_+^u}, \left(\frac{2}{c_{\ell(s),s}}\frac{r_{sj}}{K_6} \right)_{s\in S_+^u} \right\}$.

Let $\#I_b^+ = 1$. If we decrease the bid of $i_0 \in I_b^+$ by $\Delta_7 \leq \min\left\{ b_j^{i_0}, \varepsilon, 1 \right\}$, we obtain $(\Delta m_0) = \frac{Q_j^{-i_0}+\varepsilon}{Q_j+\varepsilon}\Delta_7 \geq \frac{\varepsilon}{Q_j+\varepsilon}\Delta_7$ and[18] $\left(\Delta\theta_j^{i_0} \right) = -\frac{1}{\pi_j^\varepsilon}\cdot\frac{\varepsilon}{b_j^{i_0}+\varepsilon-\Delta_5}\Delta_7 \geq -\frac{1}{\pi_j^\varepsilon}\cdot\frac{2\varepsilon}{b_j^{i_0}+\varepsilon}\Delta_7$. By converting this decrease in the asset holding into a change in future money holding by taking care to bankruptcy, we obtain after a tedious computation that :

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[17] Since $\frac{q_j^{i_0}}{Q_j} \leq \frac{1}{2}$ we can say that $Q_j^{-i_0} \geq \frac{Q_j}{2}$. It follows that $\frac{Q_j^{-i_0}+\varepsilon}{Q_j+\varepsilon}\Delta \geq \frac{\frac{Q_j}{2}+\varepsilon}{Q_j+\varepsilon} \geq \frac{\frac{1}{2}(Q_j+2\varepsilon)}{Q_j+2\varepsilon}$.

[18] Let us first observe that $\left(\Delta\theta_j^{i_0} \right) \geq -\frac{1}{\pi_j^\gamma}\frac{2\varepsilon}{b_j^{i_0}+2\varepsilon-\Delta_5}\Delta_7$ because $\frac{-\varepsilon}{b_j^{i_0}+\varepsilon}$ decreases with ε. Moreover since $\Delta_7 \leq \varepsilon$, it follows $\Delta m_s \geq -\frac{1}{\pi_j^\gamma}\frac{2\varepsilon}{b_j^{i_0}+\varepsilon}\Delta_7$

$$\Delta\left(x^{i_0}, m^{i_0}\right) \geq \frac{\varepsilon \Delta_7}{\left(Q_j+\varepsilon\right)} \cdot\left(-2\left(\sum_{s \in S_+^u} \frac{r_{sj}}{\left(\pi_j^{\varepsilon}\right)^2} e_s+\sum_{s \in S_+^b} \frac{1}{p_{\ell(s), s}^{\varepsilon}} \frac{r_{sj}}{\left(\pi_j^{\varepsilon}\right)^2} e_{\ell(s), s}\right)+e_0\right)$$

and construct $K_j^3=\max \left\{\sqrt{\frac{2}{\delta_7}}\left(\sum_{s \in S_+^u} r_{sj}^2+\sum_{s \in S_+^b}\left(\frac{r_{sj}}{c_{\ell(s), s}}\right)^2\right)^{\frac{1}{4}},\left(\sqrt{\frac{2 r_{sj}}{K_7}}\right)_{s \in S_+^u},\left(\sqrt{\frac{2}{c_{\ell(s), s}} \frac{r_{sj}}{K_7}}\right)_{s \in S_+^u}\right\}$

We can therefore conclude $\pi_j^{\varepsilon} \leq K_j=\max \left\{K_j^1, K_j^2, K_j^3\right\}$

■

5.4 The main result

It now simply remains to put together the various remarks and lemmata of this section in order to prove that :

Theorem 2 *Every Shapley-Shubik game with financial markets admits at least one nice equilibrium*

Proof : Let $(\tilde{a}_{\varepsilon})$ be a css of $\varepsilon - NE$ equilibrium strategies whose limit is $(\tilde{a}_0)$. By lemmata 7 and 8, the associated sequence of commodity prices $\tilde{p}_{\varepsilon}$ admits a css of limit $\tilde{p}_0 \neq 0$. By remark 10 and the continuity of the utility functions, it follows that $\forall i=1, \ldots, I$, $U_i\left(\varphi_{\varepsilon}^i\left(\tilde{a}_{\varepsilon}\right)\right) \underset{\varepsilon \rightarrow 0}{\rightarrow} U_i\left(\varphi^i\left(\tilde{a}_0\right)\right)$ (for a ccs). Moreover, by corollary 1 and lemma 9, the asset price sequence $\tilde{\pi}_{\varepsilon}$ also admits a css of limit $\tilde{\pi}_0 \neq 0$. Hence by (i) of lemma 6 we know that $\tilde{a}_0^i \in \alpha^i(\tilde{a}_0^{-i})$. It now remains to verify that no agent has an incentive to deviate. So let us assume $\exists i$, $\exists a^i \in \alpha^i(\tilde{a}_0^{-i})$, $U_i\left(\varphi^i\left(a^i, \tilde{a}_0^{-i}\right)\right)>U_i\left(\varphi^i\left(\tilde{a}_0\right)\right)$. By (ii) of lemma 6, $\exists\left(a_{\varepsilon}^i\right)$ with $a_{\varepsilon}^i \in \alpha_{\varepsilon}^i(\tilde{a}_{\varepsilon}^{-i})$ and $a_{\varepsilon}^i \underset{\varepsilon \rightarrow 0}{\rightarrow} a_0^i$. Since, at least for $\varepsilon<1$, the allocations belong to a compact set (see proof of lemma 11), $\varphi_{\varepsilon}\left(a_{\varepsilon}^i, \tilde{a}_{\varepsilon}^{-i}\right) \underset{\varepsilon \rightarrow 0}{\rightarrow} \varphi_0$ (for a css). So, by remark 10 we know that $\varphi_0^i \geq \varphi^i\left(a^i, \tilde{a}_0^{-i}\right)$ and since the utility is non decreasing we have $U_i\left(\varphi_0^i\right)>U_i\left(\varphi^i\left(\tilde{a}_0\right)\right)$. By construction of the different ccs that we have introduced up to now and the continuity of the utility, we finally observe that $\exists \lambda>0$, for some $\varepsilon<\lambda$, $\exists a_{\varepsilon}^i \in \alpha_{\varepsilon}^i(\tilde{a}_{\varepsilon}^{-i})$, $U_i\left(\varphi_{\varepsilon}^i\left(a_{\varepsilon}^i, \tilde{a}_{\varepsilon}^{-i}\right)\right)>U_i\left(\varphi_{\varepsilon}^i\left(\tilde{a}_{\varepsilon}\right)\right)$. But this contradict the fact that $\tilde{a}_{\varepsilon}$ is a $\varepsilon - NE$.

■

6 Conclusion

In this paper we have extended the Shapley-Shubik model to a two period economy with financial markets. In fact, we have considered a two period and S-state economy in which the agents have the opportunity to buy in the first period a certain number of *numéraire* assets in order to reallocate their purchasing power between both the two periods and the future S states. In this *numéraire asset* economy *à la* Geanakoplos-Polemarchakis (1986), we simply replace, on each market, the standard walrasian market mechanism by the one depicted by Shapley & Shubik (1977). Since in this last case, the agents make their choice under cash-in-advance constraints, the financial asset give them the opportunity, as usually in a GEI economy, to reallocate partially these constraints across periods and states. Within this context, we were mainly concern by the existence issue and especially the existence on nice equilibria, i.e. equilibria where each trading post is active.

This is why several topics concerning these equilibria remain of our agenda. Concerning the existence issue, one can, for instance, try to introduce *real* assets instead or *numéraire* ones. This clearly modifies the nature of the equilibrium because assets do not, in this case, contribute to a reorganization of the different cash-in-advance constraints. It modifies the amount of commodities that an agent owns in the second period and therefore changes her capacity to offer commodities. If one maintains, in this case, the assumption that the agents make their choice under some cash-in-advance constraints, the existence result could be obtained without a secured lending assumption since each player cannot buy to much assets due to her first period cash-in-advance constraint and sell to much asset due to her limited ability to buy back commodities in the second period.

But we must concede that the cash-in-advance and/or the secured leading constraints exclude the possibility of arbitrary short sells on the financial markets. From that point of view, it is quite clear that the question of the asymptotic convergence toward a walrasian GEI economy as the number of players increases leads to negative conclusion. As noted by Amir, Sahi, Shubik & Yao (1990), this simply follows from the idea that liquidity constraints, who have no role to play in general equilibrium play a fundamental role in this kind of strategic market games. In the same vein, and following in some sense Koutougeras (2003), we may even conjecture that arbitrage opportunity remains on the financial markets, even at the limit, since the secured lending assumption avoid unlimited short sells. But *a contrario*, if these constrains are not binding for enough agents one may expect that the arbitrage opportunities disappear as the economy becomes large.

Finally, one can tries to look at a model in which the cash-in-advance and/or the secured leading constraints can be excluded. In the first case it can be interesting to move to a strategic market game with fiat money in the spirit of Peck, Shell & Spear (1992). In this setting, one typically considers a strategic market game with financial asset (instead of nominal or real assets) without any cash-in-advance constraint. But the question of existence becomes again an open issue. In the second case, one can try to substitute to the secured lending constraint a mechanisms in which default is allowed and some punishments are implemented (see Dubey Geanakoplos & Shubik 2005). But in this last case, the fact that the agents are able to manipulates this new constraints remains again an open issue.

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APPENDIX

Lemma 10 *Let $U : \mathbb{R}_+^{(L+1)S+1} \to \mathbb{R}$ be a continuous increasing and quasi-concave function, let e_i be the i^{th} vector of the canonical basis of $\mathbb{R}^{(L+1)S+1}$, and let $(\bar{x}, \bar{m}) \in \mathbb{R}_+^{(L+1)S+1}$. In this case $\exists\delta > 0$ with the property that if $\|z\| \leq \delta$ and $(\bar{x}, \bar{m}) + z \geq 0$ then $\forall\lambda \in]0, 1[$, $u^i((\bar{x}, \bar{m}) + \lambda(z + e_i)) > u^i(\bar{x}, \bar{m})$*

Proof : Let $\overline{B_\rho}((\bar{x}, \bar{m}))$ be a closed ball around $(\bar{x}, \bar{m})$ with finite radius ρ and let us construct

$$D = \left\{x \in \mathbb{R}_+^{(L+1)S+1} : \|(x, m)\| \leq \|(\bar{x}, \bar{m})\| + \rho\right\}$$

Observe that (i) $\Delta := \min_{y \in \overline{B_\rho}((\bar{x}, \bar{m}))} (U(y + e_i) - U(y)) > 0$ because U is increasing and (ii) the restriction of the utility function to the compact set D is, by Heine's theorem, a uniformly continuous function. It follows by (ii) that $\forall\varepsilon > 0$, $\exists\delta > 0$ with the property that $\forall y \in B_\delta((\bar{x}, \bar{m})) \cap D$, $|U((\bar{x}, \bar{m})) - U(y)| < \varepsilon$. Now set $\varepsilon = \Delta$ and choose, if necessary, $\rho \geq \delta$. Because U is increasing and uniformly continuous on D, we can assert that $\forall y \in B_\delta((\bar{x}, \bar{m})) \cap D$, $U((\bar{x}, \bar{m})) < U(y + e_i)$. But y can be written as $y := (\bar{x}, \bar{m}) + z$ with $\|z\| < \delta$ and $(\bar{x}, \bar{m}) + z \geq 0$. It follows that if $\|z\| \leq \delta$ and $(\bar{x}, \bar{m}) + z \geq 0$ then $U((\bar{x}, \bar{m}) + z + e_i) > U((\bar{x}, \bar{m}))$. Finally, observe that U is quasi-concave, and conclude that $\forall\lambda \in]0, 1[$, $U((\bar{x}, \bar{m}) + \lambda(z + e_i)) > U(\bar{x}, \bar{m}))$.

■

Lemma 11 *Under the boundary assumption, there exists $K > 0$ with the property that for every agent and every allocation (x^i, m^i) of agent i which is both reachable and individually rational we have $(x^i, m^i) \gg Ke$ with $e := (1, \ldots, 1) \in \mathbb{R}_+^{(L+1)S+1}$*

Proof : Let us first observe that the feasible allocation set in a $\varepsilon - game$ is given by

$$F_\varepsilon := \left\{ \begin{array}{c} (x^i, m^i)_{i=1}^I \in \mathbb{R}_+^{((L+1)S+1)I} : \forall \ell s \quad \sum_{i=1}^n \left(x_{\varepsilon,\ell s}^i(a) - \omega_{\ell s}^i\right) = \frac{\varepsilon(B_{\ell s} - Q_{\ell s})}{B_{\ell s} + \varepsilon}, \\ \sum_{i=1}^n \left(m_{\varepsilon,0}^i(a) - \mu_0^i\right) = \sum_{j=1}^J \frac{\varepsilon(Q_j - B_j)}{Q_j + \varepsilon} \\ \text{and } \forall s = 1, \ldots, S \;\; \sum_{i=1}^n \left(m_{\varepsilon,s}^i(a) - \mu_s^i\right) = \sum_{j=1}^J r_{s,j} \left(\frac{\varepsilon(B_j - Q_j)}{B_j + \varepsilon}\right) + \sum_{\ell=1}^L \frac{\varepsilon(Q_{\ell s} - B_{\ell s})}{Q_{\ell s} + \varepsilon} \end{array} \right\}$$

But for $\varepsilon < 1$ and for each item of trade we have $\frac{\varepsilon(B-Q)}{B+\varepsilon} \leq 1$ and $\frac{\varepsilon(Q-B)}{Q+\varepsilon} \leq 1$. We can therefore assert that the set of feasible allocations is included in :

$$F_{\max} := \left\{ \begin{array}{c} (x^i, m^i)_{i=1}^I \in \mathbb{R}_+^{((L+1)S+1)I} : \forall \ell s \ \sum_{i=1}^n \left(x_{\ell s}^i - \omega_{\ell s}^i\right) \leq 1,\ \sum_{i=1}^n \left(m_0^i - \mu_0^i\right) \leq J \\ \text{and } \forall s = 1, \ldots, S \ \sum_{i=1}^n \left(m_s^i - \mu_s^i\right) \leq L + \sum_{j=1}^J r_{s,j} \end{array} \right\}$$

which is a non-empty compact set. Now let us denote by $F_{\max}^i$ the projection of $F_{\max}$ on agent i's consumption set. Since the set R^i of individually rational allocation is closed and belong to $\mathbb{R}_{++}^{(L+1)S+1}$. It follows that $\exists K_{\ell,s}^i > 0$ with the property that

$$K_{\ell,s}^i = \min_{(x,m) \in R^i \cap F_{\max}^i} \|(x,m) - (0, \ldots, 0, x_{\ell,s}, 0, \ldots, 0)\| > 0$$

Since the same argument not only holds if we replace $x_{\ell,s}$ by m_s and can also be replicated for each agent. We can set $K = \min_{\ell=1,\ldots,L, s=1,\ldots,S, i \in I} \left\{K_{\ell s}^i, K_s^i\right\}$ and conclude.

■