The Law of Indifference, Equilibrium, and Equilibration in Jevons, Walras, Edgeworth, and Negishi

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1 Introduction

In the glorious decade running from 1871 to 1881, Jevons, Walras, and Edgeworth laid the foundations of most modern microeconomic analysis: in his *Theory of Political Economy* (1^{st} edition 1871, 2^{nd} edition 1879; henceforth *TPE*), not only did Jevons formalise for the first time, by means of his celebrated "equations of exchange", the conditions defining the equilibrium of a pure-exchange two-commodity, two-trader economy, but he also opened up a line of thought that would lead, about one century later, to the emergence of a theory of decentralised exchange based on pairwise interactions (see, e.g., Kunimoto and Serrano, 2004); in a series of *mémoires* published in the mid-1870s (1874, 1876a, 1876b, 1877a), as well as in the 1^{st} edition of his *Eléments d'économie politique pure* (henceforth *EEPP*), appearing in two instalments in the same stretch of time (1874-77), Walras started off a grand theoretical approach, subsequently named 'general equilibrium theory', applying not only to exchange economies with an arbitrary number of markets and agents, but also to multi-market and multi-agent economies with production and capital formation; finally, in his *Mathematical Psychics* (1881; henceforth *MP*), Edgeworth sealed the revolutionary decade 1871-1881 by initiating a tradition that would lead, about eighty years later, to the development of a theory of decentralised exchange based on multilateral interactions, a special branch of the broader approach known as 'coalitional game theory'.

All the three founding fathers made use of at least one equilibrium concept in their respective theoretical systems, occasionally naming it "competitive

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equilibrium". Among the equilibrium notions respectively employed by Jevons, Walras, and Edgeworth there obviously exist important relations, sometimes explicitly recognised by their very inventors. Yet, there also exist important differences among the behavioural assumptions and equilibrium concepts underlying the various approaches. Moreover, in relatively recent times, the expressions "competitive equilibrium" and "competitive behaviour" have started to be frequently used to denote the specifc Walrasian varieties of equilibrium and behaviour. Hence, in the following, we shall avoid using ambiguous expressions such as "competitive" to refer to notions characteristic of the various theoretical systems, identifying instead the concepts pertaining to each approach by means of specific qualifiers (such as Jevonsian, Walrasian, or Edgeworthian).

In developing their equilibrium theories the three economists relied upon some variety of a so-called 'Law', which nowadays is commonly referred to as the 'Law of Indifference' or the 'Law of One Price'. Yet, once again, the interpretation and use of the 'Law' were quite diverse among the three; hence, to employ a common term to denote such different concepts might be misleading. In the sequel, therefore, we shall abide to the following convention: the expression 'Law of Indifference' will be reserved to denote the concept as employed by Jevons, who, after all, was the one who invented that label (in the 2^{nd} edition of *TPE*, for in the first edition he had made use of the alternative expression "principle of uniformity"); the expression 'Law of One Price' will be used, instead, to denote the concept as employed by Walras, for the underlying idea is often regarded as the hallmark of Walrasian economics (see, e.g., Koutsougeras 2003); finally, the peculiar reinterpretation of Jevons's Law of Indifference suggested by Edgeworth (1881) will be qualified as 'Edgeworthian Law of Indifference'.

The Law of Indifference plays a fundamental role in both the derivation of Jevons's "equations of exchange" and the characterisation of his equilibrium concept. Being aware of the irreplaceable position of the Law in his theoretical system, Jevons tried to provide some empirical justification for it, with the only unintended result of turning the Law into a quasi-tautology and, at the same time, of seriously restricting the scope of his equilibrium theory. Walras's expanded the range of application of his Law of One Price, which was no longer confined, as in Jevons, to equilibrium analysis only, but was so conceived as to embrace Walras's equilibration analysis too, namely, his celebrated *tâtonnement* construct. Not differently from Jevons, also Walras would have liked to empirically or theoretically support his use of the Law; a promising candidate for this purpose would have appeared to be Cournot's theory of arbitrage, a theory well-known to Walras and actually employed by him to explain the emergence of a consistent price system in a multi-commodity economy; and yet, despite his acquaintance with Cournot's theory, Walras proved unable to use it to buttress his use of the Law of One Price, so that the latter's status within the Walrasian theoretical system remained that of a postulate lacking theoretical underpinnings. Finally, Edgeworth resumed Jevons's original idea of strictly confining the reach of the Law of Indifference to equilibrium analysis, severely censuring Walras's suggested extension of the scope of his Law of One Price to the sphere of equilibration analysis too. Unlike Jevons, however, Edgeworth did provide a theoretical foundation for the Law of Indifference, making it emerge as a limiting equilibrium result, produced by the joint operation of the Edgeworthian replication and recontracting mechanisms and associated with the unbounded increase in the size of the economy.

Almost exactly one century after Edgeworth's achievements, Negishi, in the unconventional paper "A Note on Jevons's Law of Indifference and Competitive Equilibrium" (1982), goes directly back to Jevons's Law of Indifference. In his paper Negishi takes side with both Jevons's and Edgeworth's original interpretation of the Law of Indifference as an equilibrium property, while disagreeing with Walras's attempted extension of the scope of his Law of One Price to the equilibriation domain. Yet, parting company with Jevons, Negishi tries to prove that the arbitrage mechanism underlying Jevons's Law of Indifference can be brought to light and easily formalised by means of a simple reinterpretation and extension of Edgeworth's concept of coalition and the associated recontracting mechanism; then, parting company also with Edgeworth, he tries to prove that such mechanism, far from requiring the coexistence of infinitely many traders in the economy, is at work and fully effective in any finite economy, even a very small one, provided that there is some room for arbitraging (which implies that the traders' number must be greater than two, a requirement apparently neglected by Jevons). Negishi's remarkable findings, substantially ignored for almost two decades and approvingly resumed by a few scholars only at the turn of the century (see Rebeyrol, 1999, and Pignol, 2000), have not yet been thouroughly assessed in the literature. The aim of the present paper is fill this gap: first, by reconstructing the various uses of the Law of Indifference and its variants in the diverse theoretical traditions originated by Jevons, Walras, and Edgeworth; and, secondly, by critically discussing Negishi's attempt to revive Jevons's Law of Indifference as a fully microfounded and extremely powerful equilibrium property of finite competitive economies where arbitraging activities are allowed for.

The paper is structured as follows. Since most of the subsequent discussion will be couched in terms of the simplest model actually employed by Jevons, Walras, Edgeworth, and Negishi to develop their respective ideas about the Law of Indifference and its variants, namely, the model of a pure-exchange two-commodity economy with cornered traders, in Section 2 we shall first of all introduce the conceptual apparatus (both analytical and diagrammatical) required for our later purposes; then, by using that conceptual machinery, we shall examine the role played by the Law of Indifference in the construction of Jevons's theory of exchange and, in the light of these findings, we shall investigate the nature of the Jevonsian equilibrium concept. In Section 3, after discussing Walras's extension of the range of application of the Law of One Price from equilibrium to equilibration analysis, we shall scrutinize the nature of the Walrasian equilibrium concept and, finally, we shall explain Walras's inability to microfound the Law of One Price by means of the Cournot-Walras or any other theory of arbitrage. In Section 4, after discussing Edgeworth's reinterpretation of Jevons's Law of Indifference and equilibrium concept as properties of unboundedly large economies, we shall investigate his attempts at theoretically supporting both constructs by means of the Edgeworthian recontracting and replication mechanisms. In Section 5 we shall critically discuss Negishi's conjecture that no large number of traders is required to elicit price uniformity and competive equilibrium, provided that arbitrage opportunities are allowed for and duly taken into account. Section 6 concludes.

2 Jevons on the Law of Indifference, equilibrium and equilibration

Jevons's theory of exchange is developed in Chapter 4 of *TPE*, especially in its three central Sections which, in the second edition of *TPE*, are called "The Law of Indifference", "The Theory of Exchange", and "Symbolic Statement of the Theory", respectively[1]. In the first of these three Sections Jevons introduces the Law of Indifference as "a general law of the utmost importance in economics", adding the somewhat curious qualification that what it asserts "is undoubtedly true, with proper explanations"[2]. The Law is defined by means of the following italicised sentence:

> *In the same open market, at any one moment, there cannot be two prices for the same kind of article.* (Jevons 1970, p. 137)

As can be seen, the very statement of the Law raises the question of the time structure of the analysis. For,

> though the price of the same commodity must be uniform at any one moment, it may vary from moment to moment, and must be conceived in a state of continual change. Theoretically speaking, it would not usually be possible to buy two portions of the same commodity *successively* at the same ratio of exchange [...]. Strictly speaking, the ratio of exchange at any moment is that of dx_2 to dx_1, of an infinitely small quantity of a commodity to the infinitely small quantity of another which is given for it. The ratio of exchange is really a differential coefficient. (Jevons 1970, pp. 137-138[3])

[1] Jevons, 1970, pp. 136-144. In the following, we shall chiefly refer to the 1970 edition of *TPE* (Jevons 1970), edited and introduced by R.D. Collison Black, which can be easily retrieved. The text of *TPE* in the 1970 edition is the same as that of the fourth edition (Jevons 1911), which, in turn, is based on the text of the second edition, as slightly revised and edited by Jevons's son, H.S. Jevons. The first edition (1871) will be explicitly referred to only when the differences with the second and the following ones are theoretically or historiographically relevant.

[2] See Jevons 1970, p. 137. The meaning of the qualification will become clear in the following. It is worth noting that, in the first edition of *TPE*, the same "law" had been labelled as the "principle of uniformity" (see Jevons 1871, p. 99).

[3] In order to adopt a uniform notation throughout the paper, in the above quote we have replaced the symbols originally employed by Jevons with other equivalent symbols. When necessary, and without further notice, we shall repeat the same procedure both in the remainder of this Section and in the following Sections of the paper. Occasionally, we shall also slightly

Jevons's methodological stance, as it emerges from the above passage, would seem to naturally lead him towards a continuous-time dynamical treatment of the exchange problem, patently modelled upon the pattern of rational mechanics, a treatment where all the relevant variables - traded quantities of commodities and respective rates of exchange in the exchange problem; positions, velocities, and accelerations of material points in mechanics - are viewed as continuous functions of time. Yet, an entirely different model is eventually provided by Jevons, who justifies his surprising change of theoretical perspective as follows:

> We must carefully distinguish [...] between the statics and the dynamics of this subject. The real condition of industry is one of perpetual motion and change. [...] If we wished to have a complete solution of the problem in all its natural complexity, we should have to treat it as a problem of motion - a problem of dynamics. But would it surely be absurd to attempt the more difficult question when the more easy one is yet so imperfectly within our power. It is only as a purely statical problem that I can venture to treat the action of exchange. Holders of commodities will be regarded not as continuously passing on these commodities in streams of trade, but as possessing certain fixed amounts which they exchange until they come to equilibrium. (Jevons 1970, p. 138)

Hence, by taking a purely "statical view of the question", Jevons is eventually able to put forward his equilibrium model. The formal model deals with two traders, called "trading bodies". Jevons's "trading body" is an elusive concept, for which the following loose definition is offered:

> By a *trading body* I mean, in the most general manner, any body either of buyers or sellers. The trading body may be a single individual in one case; it may be the whole inhabitants of a continent in another; it may be the individuals of a trade diffused in a country in a third. (Jevons, 1970, p. 135)

So Jevons's "trading body" can be either an individual decision maker, in conformity with the standard contemporary interpretation of the concept of a trader, or an aggregate of individuals. But, in the second case, Jevons oscillates between two alternative stances: on the one hand, he would like to endorse a strictly reductionistic position, as when he states that "the law, in the case of the aggregate, must depend on the fulfilment of law in the individuals", so that the behaviour of the aggregate can be exactly deduced from the laws ruling the behaviour of the individuals composing it (Jevons 1970, p. 135); on the other, he is forced to recognise that, barring exceptional cases, such as the case obtaining when all the individuals belonging to an aggregate are identical, "the average

modify a few assumptions originally made by Jevons, Walras, or Edgeworth, in order to simplify the exposition or to correct minor slips in their writings. No such replacement or change, however, will entail any substantive alteration of their original statements or conclusions.

laws applying to [the aggregates] do not pretend to represent the character of any existing thing" (Jevons 1970, p. 136; see also p. 86).

This ambiguity is never dispelled by Jevons. Though being unable to specify whether the laws of an aggregate can be formally deduced from the laws of the individuals composing it, he never gives up the interpretation of a "trading body" as an aggregate of individuals. Yet, in dealing with the formalized part of his theory of exchange, he invariably interprets the two "trading bodies" appearing in it as if they were two individual decision units - a stance that we shall adopt in the following. The fact is that, for reasons that will become clear in a while, Jevons needs both interpretations, even if he is forced by the requirements of the theory to treat his "trading bodies" as individual decision makers.

Let us then consider a pure-exchange economy with a finite number $L = 2$ of commodities, denoted by $l = 1, 2$, and a finite number $I = 2$ of consumers-traders (henceforth indifferently referred to as either consumers or traders), denoted by $i = A, B$. Each consumer i is characterized by a consumption set $X_i = \{x_i \equiv (x_{1i}, x_{2i})\} = \mathbb{R}^2_+$, a utility function $u_i : X_i \to \mathbb{R}$, and endowments $\omega_i \equiv (\omega_{1i}, \omega_{2i},) \in \mathbb{R}^2_+ \setminus \{0\}$. Let $x = (x_A, x_B) \in X = \times_{i=A,B} X_i \subset \mathbb{R}^{2\times 2}_+$ be an allocation; $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2) = \sum_{i=A,B} \omega_i \in \mathbb{R}^2_{++}$ be the aggregate endowments; $A^{2\times 2} = \left\{ x \in X \mid \sum_{i=A,B} x_i = \bar{\omega} \right\}$ be the set of feasible, non-wasteful allocations. A pure-exchange, two-commodity, two-consumer economy with the above characteristics will be called an Edgeworth Box economy, so named by Bowley (1924) after Edgeworth (1881), and will be denoted in the following by $\mathcal{E}^{2\times 2}_J = \left\{ \left(\mathbb{R}^2_+, u_i(\cdot), \omega_i \right)_{i=A,B} \right\}$, where the subscript J stands for Jevonsian. The consumers' characteristics (consumption sets, utility functions, endowments) represent the data of the Edgeworth Box economy; they are assumed to be fixed up until the exchange problem is solved and an equilibrium is established. The period over which the data remain fixed will be referred to as a 'trade round'. In conformity with standard usage in contemporary models of an Edgeworth Box economy, the traders' utility functions are assumed to be continuously differentiable, strongly monotonic, and strictly quasi-concave. Further, in accordance with Jevons's original assumptions, the traders are supposed to be cornered, that is, to hold a positive quantity of one commodity only; specifically, in the following we shall assume: $\omega_A = (\omega_{1A}, \omega_{2A}) = (\bar{\omega}_1, 0)$ and $\omega_B = (\omega_{1B}, \omega_{2B}) = (0, \bar{\omega}_2)$.

Jevons's modelling choices have far-reaching consequences on the time structure of the analysis. Probably Jevons is not fully aware of them; certainly he does not spell them out in full. Yet, it is convenient to make them explicit right now, for they affect the entire discussion: the evolution of the economy $\mathcal{E}^{2\times 2}_J$ over continuous time must be viewed as a sequence of disconnected, discrete-time trade rounds, each characterised by its own data and, hopefully, its own equilibrium; since each trade round must be viewed as self-contained, there cannot be any carry-over of endowments from one trade round to the next; at the same time, since the data must not change over each trade round, the endowments must not wear off or be consumed up until an equilibrium is reached; so that, in the end, the endowments must be perfectly durable over a trade round and perfectly perishable when the trade round is over (see Hicks 1989, pp. 7-11).

The above specified model of an Edgeworth Box economy can be graphically represented by means of the homonymous diagram, where the lengths of the sides of the rectangle are respectively given by the aggregate endowments, $\bar{\omega}_1$ and $\bar{\omega}_2$, and each point in the rectangle represents a feasible, non-wasteful allocation $x = (x_A, x_B)$ (see Figure 1 below).

Figure 1 about here

The Edgeworth Box model and diagram conform to Jevons's conception of the "statical view" of the exchange problem; specifically, they abide by the "statical" rule that the "holders of commodities [must] be regarded [...] as possessing certain fixed amounts which they exchange until they come to equilibrium". As a matter of fact, the Edgeworth Box diagram can only be plotted if it is assumed that the traders' characteristics, including their holdings of commodities, are fixed and unchanging over the time period required for the exchange process to take place, that is, over the period that has been called a 'trade round'.

Now, since, under the "statical" rule, the traders are explicitly imagined to "exchange until they come to equilibrium", it would appear that, according to Jevons, the analysis of the equilibration process is not inconsistent with the "statics of the subject", provided that the data of the economy are not allowed to change during the process. Yet, this 'liberal' interpretation of the "statical" method is immediately disavowed by Jevons himself, as the following passage shows:

> It is much more easy to determine the point at which a pendulum will come to rest than to calculate the velocity at which it will move when displaced from that point of rest. Just so, it is a far more easy task to lay down the conditions under which trade is completed and interchange ceases, than to attempt to ascertain at what rate trade will go on when equilibrium is not attained.
>
> The difference [between the statics and the dynamics of the exchange problem] will present itself in this form: dynamically we could not treat the ratio of exchange otherwise than as the ratio of dx_2 and dx_1, infinitesimal quantities of commodity. Our equations would then be regarded as differential equations, which would have to be integrated. But in the statical view of the question we can substitute the ratio of the finite quantities x_2 and x_1. (Jevons, 1970, p. 138)

The first paragraph in the above quote is relatively clear: by relying on the well-known mechanical analogy of the motion and rest of a pendulum, Jevons means to suggest that the "statical" theory of exchange ought to refrain from studying the complicated equilibration process (likened to the motion of a pendulum), in order to exclusively focus on the much easier task of characterising an already achieved equilibrium position (likened to the position of rest of a pendulum). So, in the end, contradicting his immediately preceding stance on the same issue, Jevons apparently concludes that the analysis of the equilibration process ought to be banned from the "statics of the subject"[4].

The second paragraph in the quote is more cryptic. In order to elucidate its meaning, a few further concepts are needed. Let us then consider what Jevons calls an "act of exchange" between the two traders (Jevons, 1970, pp. 138-9). Such an "act" involves the trade of "infinitely small" or "finite" quantities of the two commodities: it will be called 'differential' in the former case and 'finite' in the latter. In either case the quantity of the commodity given in exchange will be taken to be negative (that is, $dx_{li} < 0$ or $\Delta x_{li} < 0$, if commodity l is given by trader i, for $i = A, B$ and $l = 1, 2$), while the quantity of the commodity received in exchange will be taken to be positive (that is, $dx_{li} > 0$ or $\Delta x_{li} > 0$, if commodity l is received by trader i, for $i = A, B$ and $l = 1, 2$). Since an "act of exchange" is necessarily bilateral, the vectors of the quantities traded satisfy the following conditions: $(dx_{1A}, dx_{2A}) = -(dx_{1B}, dx_{2B})$, if the "act" is 'differential'; $(\Delta x_{1A}, \Delta x_{2A}) = -(\Delta x_{1B}, \Delta x_{2B})$, if the "act" is 'finite'. Finally, let $dx_l = |dx_{li}|$ (resp., $\Delta x_l = |\Delta x_{li}|$) be the absolute value of the quantity exchanged of commodity l in a 'differential' (resp., 'finite') "act of exchange". Then, following Jevons (1970, pp. 138), a ratio of the type $\frac{dx_2}{dx_1} = \left|\frac{dx_{2A}}{dx_{1A}}\right| = \left|\frac{dx_{2B}}{dx_{1B}}\right| > 0$ (resp., $\frac{\Delta x_2}{\Delta x_1} = \left|\frac{\Delta x_{2A}}{\Delta x_{1A}}\right| = \left|\frac{\Delta x_{2B}}{\Delta x_{1B}}\right| > 0$) will be called a 'differential' (resp., 'finite') "ratio of exchange". As we shall see, a special kind of 'finite' "ratio of exchange" plays a fundamental role in Jevons's theory of exchange: it is the 'finite' "ratio" $\frac{x_2}{x_1}$, where $x_1 = -\Delta x_{1A} = -(x_{1A} - \omega_{1A}) = \bar{\omega}_1 - x_{1A} = \Delta x_{1B} = x_{1B} - \omega_{1B} = x_{1B}$ and $x_2 = -\Delta x_{2B} = -(x_{2B} - \omega_{2B}) = \bar{\omega}_2 - x_{2B} = \Delta x_{2A} = x_{2A} - \omega_{2A} = x_{2A}$.

Going back now to the second paragraph, a tentative interpretation may run as follows. If the theorist were to squarely face the equilibration issue, allowing for "acts of exchange" to take place "successively" over the 'trade round', she would be forced to recognise not only that "the price of the same commodity [...] may vary from moment to moment, and must be conceived in a state of continual change", but also that "the ratio of exchange at any moment is that of dx_2 to dx_1, of an infinitely small quantity of a commodity to the infinitely small quantity of another which is given for it. The ratio of exchange is really a differential coefficient"[5]. Under such assumption, the theorist would

[4] Two decades after Jevons's turnabout, the 'liberal' interpretation of the "statical" method, allowing for the study of the equilibration process under the assumption of constancy of the data, will be enthusiastically endorsed by Bortkiewicz (1891, p. 359) in his defence of Walras's analysis of the equilibration process, the *tâtonnement* construct, against Edgeworth's strictures (1889a and 1889b, pp. 276-7): for Bortkiewicz, in fact, Jevons's 'liberal' interpretation exactly coincides with the stance taken by Walras in his analysis of *tâtonnement* in exchange. Jevons's ambiguous and self-contradictory statements in the above passages are pointed out by Edgeworth in his reply to Bortkiewicz (1891a, p. 366, fn.1). Edgeworth's conclusion on this point, however, is that Jevons's most authentic and final stance on the interpretation of the "statical" method is that the analysis of the equilibration process ought to be remorselessly left out of "statics".

[5] It should be noted that, contrary to what Jevons appears to suggest, the two statements are really disconnected: for the statement that a "price" or a "ratio of exchange" "must be

find herself in the condition of explaining an equilibration trajectory such as the one, plotted in Fig. 1, from the endowment allocation ω to the putative equilibrium allocation x^d, where the superscript d stands for "dynamical". But, in order to provide such explanation, the theorist ought to rely on a dynamical theory of the equilibration process, that is, she ought to write down a system of "differential equations" governing the process and to solve it. Yet, according to Jevons, this undertaking is far beyond the reach of current economic theory. Hence, the only way out is to discard the "dynamical" perspective in favour of the "statical" view, which also means to desert the equilibration issue in favour of the equilibrium characterisation issue, or else, what is the same for Jevons (given his unwarranted identification of equilibration "acts of exchange" with 'differential' "acts of exchange"), to get rid of the differential coefficients dx_2 and dx_1, together with the differential equations potentially associated with them, in favour of the finite quantities x_2 and x_1, together with the ordinary equations potentially associated with them.

Let us then examine how Jevons faces and solves the equilibrium characterisation issue. In order to tackle this question, however, we need to introduce one further concept, which will prove useful in the following discussion not only of Jevons's approach, but also of Walras's, Edgeworth's, and Negishi's theoretical models. Given an Edgeworth Box economy satisfying the above assumptions on endowments and utilities, $\mathcal{E}_J^{2\times 2} = \left\{ \left(\mathbb{R}_+^2, u_i\left(\cdot\right), \omega_i \right)_{i=A,B} \right\}$, let us define consumer i's marginal rate of substitution of commodity 2 for commodity 1 when i's consumption is x_i, $MRS_{21}^i(x_i)$, as the quantity of commodity 2 that consumer i would be willing to exchange for one unit of commodity 1 at the margin, in order to keep his utility unchanged at the original level $u_i(x_i)$. From this definition it follows that:

$$MRS_{21}^i(x_i) \equiv \left| \frac{dx_{2i}}{dx_{1i}} \right|_{u_i(x_i+dx_i)=u_i(x_i)} = \frac{\frac{\partial u_i(x_i)}{\partial x_{1i}}}{\frac{\partial u_i(x_i)}{\partial x_{2i}}},\ i = A, B.$$

In their writings Jevons, Walras, and Edgeworth ignore the notion of the marginal rate of substitution. Yet, they do know and systematically employ the notion of the marginal utility of commodity l for consumer i, which, under the

conceived in a state of continual change" does not entail the statement that "the ratio of exchange at any moment [...] is really a differential coefficient". As a matter of fact, the idea that the "ratio of exchange" is continuously changing as a function of time has nothing to do with the idea that an "act of exchange" at any instant must be of the 'differential' type: for a 'finite' "act of exchange" may well be instantaneously carried out. Jevons's confusion between these two aspects is probably due, at least in part, to the misleading influence wielded on his theoretical system by the mechanical analogies pervading all his system of thought: for while the motion of a material point is continuously differentiable with respect to time and exhibits no "jumps" in (ordinary) space, "acts of exchange" may well be continuous in time, but are by no means restricted to exclusively give rise, at each instant, to infinitesimal displacements in the space of allocations. It is somewhat ironical that, as we shall see, Jevons himself needs to resort to a 'finite' instantaneous "act of exchange" in order to make his system of equilibrium equations determinate. This last remark suggests a further explanation for Jevons's unwarranted association between 'differential' "acts of exchange" and dynamics, an explanation to which we shall come back later.

stated assumptions on the properties of the utility functions, is well-defined and bounded away from zero everywhere in the consumption set. Moreover, though not explicitly discussing the concept of the marginal rate of substitution as such, they do implicitly make use of it in their analyses, since they compute the ratios of the values of the marginal utility functions of each consumer corresponding to specific consumption bundles and examine the role of such ratios in solving the exchange equilibrium problem.

Now, in facing the equilibrium characterisation issue, Jevons (1970, pp. 139-140) starts by asking "at what point the exchange will cease to be beneficial" for the traders. His answer is as follows: for each trader i, given an arbitrarily "established" 'differential' "ratio of exchange", say $\left|\widehat{\frac{dx_{2i}}{dx_{1i}}}\right|$, the point at which the exchange ceases to be beneficial for i, called by Jevons the "point of equilibrium", is identified by the condition that the given 'differential' "ratio" be equal to trader i's marginal rate of substitution evaluated at the "equilibrium point", $MRS^i_{21}(\hat{x}_i)$, that is:

$$\left|\widehat{\frac{dx_{2i}}{dx_{1i}}}\right| = MRS^i_{21}(\hat{x}_i) = \frac{\frac{\partial u_i(\hat{x}_i)}{\partial x_{1i}}}{\frac{\partial u_i(\hat{x}_i)}{\partial x_{2i}}},\ i = A, B. \tag{1}$$

It should be noted that Jevons, after deriving the individual optimisation conditions (1) for either trader separately (1970, p. 142), never thinks of combining them into a single equation, so that, in spite of Edgeworth's overgenerous, yet unfounded, acknowledgement of Jevons's priority (Edgeworth 1881, p. 21), he never obtains an equation of the following type:

$$MRS^A_{21}(x^C_A) = \frac{\frac{\partial u_A\left(x^C_A\right)}{\partial x_{1A}}}{\frac{\partial u_A\left(x^C_A\right)}{\partial x_{2A}}} = \frac{\frac{\partial u_B\left(x^C_B\right)}{\partial x_{1B}}}{\frac{\partial u_B\left(x^C_B\right)}{\partial x_{2B}}} = MRS^B_{21}(x^C_B), \tag{2}$$

which, together with the feasibility conditions,

$$x_A + x_B = \bar{\omega}_A + \bar{\omega}_B = \bar{\omega}, \tag{3}$$

would define the "contract curve" (or the 'Pareto set') of the Edgeworth Box economy concerned. (In Fig. 1 the "contract curve" is the curve connecting O_A and O_B.)

Now, equations (1) and (3) are not sufficient to make the model determinate, for one has just four equations to determine six unknowns $\left(x_{1A}, x_{1B}, x_{2A}, x_{2B}, \frac{dx_{2A}}{dx_{1A}}, \frac{dx_{2B}}{dx_{1B}}\right)$. It is precisely at this point that the Law of Indifference comes to rescue, for it provides the two further equations that are needed to close Jevons's model. In this regard, it is worth recalling that Jevons had concluded the section on "The Law of Indifference" with the following sentence, which came immediately after the passage, quoted in full above, on the "pendulum" analogy and the "difference" between "statics" and "dynamics":

> Thus, from the self-evident principle, stated on p. 137 [i.e., the Law of Indifference], that there cannot, in the same market, at the same moment, be two different prices for the same uniform commodity, it follows that *the last increments in an act of exchange must be exchanged in the same ratio as the whole quantities exchanged.* [...] This result we may express by stating that the increments concerned in the process of exchange must obey the equation
>
> $$\frac{dx_2}{dx_1} = \frac{x_2}{x_1}. \tag{4'}$$
>
> The use that we shall make of this equation will be seen in the next section. (Jevons 1970, p. 139; Jevons's italics)

Now, the only analytical use of the Law made by Jevons is to allow the theorist to replace the 'differential' "ratio of exchange", $\frac{dx_2}{dx_1}$, by the 'finite' one, $\frac{x_2}{x_1}$, thereby obtaining the missing equations needed to close the model. In our formalism, account being taken of (3), such equations can be written as follows:

$$\left|\frac{dx_{2A}}{dx_{1A}}\right| = \left|\frac{dx_{2B}}{dx_{1B}}\right| = \frac{dx_2}{dx_1} = \frac{x_2}{x_1} = \frac{\bar{\omega}_2 - x_{2B}}{x_{1B}} = \frac{x_{2A}}{\bar{\omega}_1 - x_{1A}}. \tag{4''}$$

By substituting (4″) into (1) and simplifying, one finally obtains:

$$\frac{\frac{\partial u_A\left(\bar{\omega}_1 - x_1^J, x_2^J\right)}{\partial x_{1A}}}{\frac{\partial u_A\left(\bar{\omega}_1 - x_1^J, x_2^J\right)}{\partial x_{2A}}} = \frac{x_2^J}{x_1^J} = \frac{\frac{\partial u_B\left(x_1^J, \bar{\omega}_2 - x_2^J\right)}{\partial x_{1B}}}{\frac{\partial u_B\left(x_1^J, \bar{\omega}_2 - x_2^J\right)}{\partial x_{2B}}}, \tag{5}$$

which are Jevons's "equations of exchange" (1970, p. 143), defining Jevons's equilibrium allocation, $x^J = \left(x_A^J, x_B^J\right) = \left(\left(\bar{\omega}_1 - x_1^J, x_2^J\right), \left(x_1^J, \bar{\omega}_2 - x_2^J\right)\right)$, and equilibrium "ratio of exchange", or relative price of commodity 1 in terms of commodity 2, $\frac{x_2^J}{x_1^J}$.

Under the stated assumptions, a Jevonsian equilibrium $\left(\frac{x_2^J}{x_1^J}, x^J\right)$ can be proven to exist, even if it is not necessarily unique. Yet, in order to simplify the exposition, let us suppose the equilibrium to be unique. Then the equilibrium "act of exchange" can be graphically represented in Fig. 1 by means of the straight line segment connecting the endowment allocation ω with Jevons's equilibrium allocation x^J. The equilibrium "ratio of exchange", in turn, is given by the absolute value of the constant slope of such straight line segment ($\tan \alpha^J$).

The peculiar use to which the Law of Indifference is put by Jevons is at the same time revealing and binding. It is revealing, for it allows the interpreter to conclusively clarify the association between 'differential' "ratios of exchange" and "dynamics", which would not appear completely intelligible otherwise (except for what has already been said in footnote 5 above). In deriving his equilibrium conditions, Jevons has to confront two distinct problems: first, he has to get rid of the equilibration process, which is still a sort of unwieldy "dynamical" process, in order to focus exclusive attention on the "statical" properties of the equilibrium allocation; secondly, he has to get rid of the two 'differential' "ratios of exchange" appearing in equations (1), what can be done by replacing them with a common 'finite' "ratio". Now, by suggesting a contrived association between 'differential' (resp., 'finite') "ratios of exchange" and "dynamics" (resp., "statics"), Jevons makes what appears to be a conceptual mistake; but, by making that mistake, he obtains the rhetorical result of mixing together his two problems, which can therefore be solved at one stroke, with the help of the Law of Indifference.

On the other hand, the use of the Law in the derivation of the equilibrium conditions severely constrains the explanatory power of Jevons's overall theory of exchange, especially of the Jevonsian equilibrium concept. First of all, as repeatedly underlined by Jevons himself (1970, pp. 133, 137, 138), the Law of Indifference holds at one specified time instant only. But then Jevons's equilibrium concept must be given an "instantaneous" interpretation too: this means that the equilibrium allocation must be imagined, as already suggested, as instantaneously reached by the two traders, by means of one single "act of exchange", taking place at one and the same equilibrium "ratio of exchange", and leading them directly from the initial endowment to the final equilibrium allocation.

The "instantaneous" interpretation of Jevons's equilibrium concept should not come as a surprise, however, since getting rid of the equilibration process and all sort of dynamical concerns was one of the chief motivations, if not the most important of all, for Jevons to assume the Law of Indifference and to operationally use it in characterising his equilibrium concept [6]. Yet, though this outcome is by no means surprising, there are at least two reasons that make it worth stressing anyway. First, Jevons himself tends occasionally to forget the "instantaneous" nature of his own equilibrium concept, for instance when he speaks of an alleged "process of exchange" underlying one instantaneous "act of exchange"[7]. Secondly, since some contemporary economists tend to think

[6] Since Jevons's equilibrium must be supposed to be "instantaneously" reached, a Jevonsian exchange economy must be conceived of as being always in equilibrium. This necessary consequence of assuming the Law of Indifference in Jevons's sense is acknowledged, e.g., by Hicks (1989, p. 7).

[7] See the sentence immediately preceding equation ($4'$) above (Jevons 1970, p. 139). In spite of this slip, however, Jevons generally appears to be well-aware of the "instantaneous" character of any single "act of exchange", hence also of the Law of Indifference as expressed by equation ($4'$), where the 'differential' ratio must *not* be taken as a function of time, at least according to the only interpretation of that equation that Jevons himself is ready to endorse.

This conclusion is indirectly confirmed by a careful perusal of the "Notes of the Lectures on Political Economy" given by Jevons at Owens College, Manchester, during the academic year 1875-6. These "Lecture Notes" were taken by Harold Rylett, a student in the course on Political Economy given by Jevons during his last year as Professor at Owens College.

that the 'bad habit' "of treating it as axiomatic that at all times the economy is in competitive equilibrium" (Hahn 1989, p. 66) is a novelty in the economics profession, it might be interesting to point out that Jevons, the very founder of the modern approach to equilibrium analysis, was already prey to such a 'bad habit' in the early 1870s.

The "instantaneous" interpretation of Jevons's equilibrium concept also affects the assumed time structure of the analysis, revealing its artificial character. As will be recalled, in order to rationalise Jevons's own assumptions, we introduced the notion of a trade round, viewed as a non-degenerate time period over which the equilibration process is allowed to take place under the assumption of data invariance. Yet, if the equilibration process collapses into one single instant, it is no longer necessary to subdivide the "perpetual motion and change" of economic life into non-degenerate periods of artificial rest, over which the data are assumed to be unchanging: when the equilibrium is "instantaneously" reached, the very notion of a trade round becomes redundant; it can only survive as a mere rhetorical device.

Forcing Jevons to adopt an "instantaneous" interpretation of the equilibrium concept, however, is not the only consequence of Jevons's interpretation and use of the Law of Indifference. Other important consequences follow from the way in which the Law is introduced and justified.

As a matter of fact, when Jevons first introduces the Law in *TPE*, without naming it, he makes some efforts to link its contents and prescriptions to the working of the market, the rules of market interaction, and the traders' characteristics, among which the traders' knowledge and motivations stand out as most important. As the following passage witnesses, when introducing the Law, Jevons apparently tries to justify it by suggesting how it can be derived from either more fundamental axioms or some basic micro-theory of individual behavior and social interaction:

> By a market I shall mean two or more persons dealing in two or more commodities, whose stocks of those commodities and intentions of exchanging are known to all. It is also essential that the ratio of exchange between any two persons should be known to all the others [...] and there must be perfectly free competition, so that anyone will

Consulted by Keynes when still in handwritten form in view of the preparation of Jevons's biographical sketch (Keynes 1936, p. 138), Rylett's Notes were later edited by R.D.C. Black and made available as Vol. VI of the *Papers and Correspondence of William Stanley Jevons* (Black 1977). While Rylett occasionally misreported or misinterpreted Jevons, his "Lecture Notes", warmly praised by Jevons himself (Black 1977, p. VIII), provide illuminating insights on Jevons's thought processes and ideas. Hence the following passage, referring to a relation equivalent to equation $(4')$ above, proves useful in understanding Jevons's own interpretation of the Law of Indifference, which is remorselessly "instantaneous":

> This is perhaps not always true. It is not really true if sales take place in succession to one another. [...] But that is not the question here, because these sales are not successive. Here we are looking at exchange which takes place *at the same time*, as it were, and in that case the whole quantities wd. be sold at the same price. (Black 1977, p. 88; italics added)

> exchange with anyone else for the slightest apparent advantage. [...] A market [...] is theoretically perfect only when all traders have perfect knowledge of the conditions of supply and demand, and the consequent ratio of exchange; and in such a market, as we shall now see, there can only be one ratio of exchange of one uniform commodity at any moment. (Jevons, 1970, pp. 133-4)

When Jevons speaks of "anyone [exchanging] with anyone else for the slightest apparent advantage", he is clearly hinting at a kind of activity, 'arbitrage', that is instrumental in bringing about price uniformity, that is, in explaining precisely that property of "perfect markets" that Jevons's Law of Indifference claims to formally express. Now, a theory of arbitrage had been put forward by Cournot in his *Recherches* a few decades before the appearance of *TPE* (1838, pp. 29-43). Yet, Jevons is apparently unable to take advantage from it: no theory of arbitrage nor, for that matter, any other theory of individual behavior is put forward in *TPE* that might help micro-founding the Law[8]. The only attempt at justifying the Law is provided by the following passage, which immediately precedes the formal definition of the Law by means of formula $(4')$:

> Suppose that two commodities are bartered in the ratio of x_1 for x_2; then every mth part of x_1 is given for the mth part of x_2, and it does not matter for which of the m-th parts. [...] We may carry this division to an indefinite extent by imagining m to be indefinitely increased, so that, at the limit, even an infinitely small part of x_1 must be exchanged for an infinitely small part of x_2, in the same ratio of the whole quantities. (Jevons 1970, p. 139)

This attempt at justifying the Law, however, is not only weak, but also utterly counterproductive: for it turns the Law into a quasi-tautology, that is, into a statement which is almost true by definition, but at the same time it deprives the Law of almost all its explanatory power and descriptive content. For Jevons's statement in the above quote applies to an individual, instantaneous "act of exchange", involving two commodities and two traders: in such a case, the rate of exchange is necessarily one and the same, however finely one might decide to partition the "act" for theoretical purposes; but, for the same reason, the implied conclusion by no means generalises to a situation where many different "acts of exchange" involving the same two commodities are simultaneously carried out by many different pairs of traders belonging to the same exchange economy. When Jevons states that the Law of Indifference "is undoubtedly true,

[8] It should be noted that Jevons became acquainted with Cournot's writings only after the publication of the first edition of *TPE*. In the second edition (1970, pp. 132-3, fn.), Jevons quotes a passage from Cournot's *Recherches* (1838, pp. 51-2, fn.) which somehow anticipates his own Law of Indifference. Yet, not even in that edition does he make any attempt to exploit Cournot's theory of arbitrage. On the contrary, and somewhat ironically, in the Preface to the second edition of *TPE*, Jevons (1970, p. 58) depicts as "not particularly useful" Chapter III of Cournot's *Recherches*, that is, precisely that chapter where Cournot's theory of arbitrage is fully developed in the context of a discussion of foreign exchanges.

with proper explanations", he has probably in mind his attempted justification the Law, which indeed turns it into a self-evident statement; but he does not realise that such quasi-tautological character would not survive in any more general context.

If Jevons were actually to use the Law of Indifference in a model of an exchange economy with more than two traders and two commodities, he would be forced to postulate it as an axiom. If this does not occur, it is only because he never really quits the two-commodity, two-trader world: his attempted extension of his theory of exchange to a multi-commodity economy ends up in a failure (Jevons 1970, pp. 152-4), while his only incursion into a world with more than two traders must rely upon his ambiguous, yet unavoidable, interpretation of the concept of a "trading body" as an aggregate of individuals. In the end, however, such severe limitations of Jevons's theory of exchange are partly explained by the fact that only in such a small world is he able to somehow corroborate his Law of Indifference.

3 Walras on the Law of One Price, equilibrium and equilibration

Walras's pure-exchange, two-commodity model is developed in the first of his published theoretical writings, the *mémoire* "Principe d'une théorie mathématique de l'échange", which appeared in 1874, as well as in the first part of Section II of the first edition of *EEPP*, whose first instalment was published in the same year[9]. Walras's two-commodity model of exchange is obviously propaedeutical to his exchange model with an arbitrary finite number of commodities, whose discussion immediately follows that of the simpler two-commodity model in *EEPP*[10]. Yet, for the purposes of this paper, it is convenient to focus attention on the two-commodity model. We shall come back to the multi-commodity model towards the end of this Section, however, in discussing Walras's theory of arbitrage.

In his pure-exchange, two-commodity model Walras studies an exchange

[9] The first edition of *EEPP* was published in two instalments, which respectively appeared in 1874 and 1877. Three further editions were published during Walras's lifetime: the second appeared in 1889, the third in 1996, and the fourth in 1900. A fifth posthumous edition, containing a few changes arranged by Walras himself before his death, appeared in 1926. In this paper all references will be made to the comparative *variorum* edition of *EEPP*, published in 1988, collating the texts of all the previous editions. When the text varies across editions and it is necessary or convenient to refer to one or more specific editions, the number(s) of the edition(s) referred to will be specified in bold, after the page number(s).

[10] In the first three editions of *EEPP* the two-commodity model was put forward in the first part (approximately the first half) of Section II, the second part of that Section being devoted to the discussion of the exchange model with more than two commodities. In the fourth and fifth editions, however, the old Section II was split into two Sections, the new Sections II and III, respectively devoted to the exchange model with only two and more than two commodities. The exchange model with an arbitrary finite number of commodities was also separately discussed in Walras's second *mémoire*, called "Equations de l'échange", which was published in 1876.

economy $\mathcal{E}_W^{2\times I} = \left\{ \left(\mathbb{R}_+^2, u_i\left(\cdot\right), \omega_i\right)_{i=1}^I \right\}$, where the subscript W stands for Walrasian, satisfying a few assumptions concerning the traders' characteristics which are similar to those already discussed in the previous Section, with one important qualification: while Jevons's model is essentially confined to the case of two traders only, Walras's model allows from the start for the participation of an arbitrary finite number of traders, so that $I \geq 2$ in this case. The set of the traders is partitioned into two subsets: the first owns $I_A \geq 1$ traders, indexed by $i_A = 1, ..., I_A$; the second owns $I_B \geq 1$ traders, indexed by $i_B = 1, ..., I_B$; therefore, the whole set owns $I = I_A + I_B \geq 2$ traders, indexed by $i = 1, ..., I_A, I_A + 1, ..., I_A + I_B = 1, ..., I$. As in Jevons, all the traders are assumed to be cornered; specifically, in the following we shall assume: $\omega_{i_A} = (\omega_{1i_A}, 0)$, with $\omega_{1i_A} > 0$, for $i = 1, ..., I_A$, and $\omega_{i_B} = (0, \omega_{2i_B})$, with $\omega_{2i_B} > 0$, for $i_B = 1, ..., I_B$, so that $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2) = (\Sigma_{i_A=1}^{I_A}\omega_{1i_A}, \Sigma_{i_B=1}^{I_B}\omega_{2i_B}) = \Sigma_{i=1}^{I}(\omega_{1i}, \omega_{2i}) \in \mathbb{R}_{++}^2$. As before, the utility functions are assumed to be continuously differentiable, strongly monotonic, and strictly quasi-concave[11]. Under these assumptions, if the number of traders is $I = 2$, then Walras's exchange economy $\mathcal{E}_W^{2\times I}$ becomes indistinguishable from Jevons's Edgeworth Box economy, $\mathcal{E}_J^{2\times 2}$, and can likewise be represented by means of an Edgeworth Box diagram; in that case we revert to the previous notation: in fact, either subset of cornered traders being a singleton, the two traders respectively belonging to the first and second subset can be simply labelled as A and B.

Now let $p = (p_{12}, 1) \in \mathbb{R}_{++}^2$ be the price system, expressed in terms of commodity 2 taken as the numeraire. Since in the economy there are only two commodities, one can only have one independent relative price, $p_{12} = \frac{1}{p_{21}}$. Walras assumes the traders to behave competitively: precisely, Walras's behavioral assumption implies that the traders take prices as given parameters and choose their optimal trade plans, called "dispositions à l'enchère" by Walras, in such a way as to maximize their respective utility functions, under their respective Walrasian budget constraints. The parametric role of prices is emphasised in the following passage:

> Si notre homme va lui-même sur le marché, il peut laisser ses dispositions à l'enchère à l'état virtuel et non effectif, c'est à dire ne déterminer sa demande d_{li} que le prix p_{12} une fois connu. Même alors ces dispositions n'existent pas moins. Mais si, par exemple, il était empêché de se rendre en personne sur le marché, ou si, pour une raison ou pour une autre, il devait donné sa commission à un ami ou ces ordres à un agent, il devrait prévoir toutes les valeurs possibles de p_{12}, depuis zéro jusqu'à l'infini, et déterminer en con-

[11] As a matter of fact, Walras typically assumes the marginal utility function of commodity l for consumer i to go to zero for $x_{li} < \infty$ (Walras 1988, pp. 107-11), thereby contradicting the strong monotonicity assumption made in the text. This assumption, as well as the others, are made here with a view to simplifying our discussion: in this way, in fact, we can dodge all boundary problems and obtain demand and supply functions that are well-defined for all positive prices. The above assumptions can anyhow be relaxed, at the cost of complicating somewhat the analysis.

séquence toute les valeurs correspondantes de d_{li}, en les exprimant d'une manière quelconque. (Walras 1988, p. 83)

For all $p = (p_{12}, 1)$, the optimization problem to be solved by trader i can be formally written as:

$$MRS_{21}^{i}(x_{1i}, x_{2i}) = \frac{\frac{\partial u_i(x_i)}{\partial x_{1i}}}{\frac{\partial u_i(x_i)}{\partial x_{2i}}} = p_{12} \qquad (6)$$

$$p \cdot x_i = p \cdot \omega_i, \qquad (7)$$

where equation (6) is Walras's "condition de satisfaction maxima" of trader i and equation (7) is the Walrasian budget constraint of the same trader, for $i = 1, ..., I$ (Walras 1988, pp. 111-7).

By solving this system, one obtains for each trader the individual gross demand functions for both commodities, from which one can derive the individual net demand and supply functions for either commodity, that is:

$$\begin{aligned} x_{i_A}(p_{12}) &= (x_{1i_A}(p_{12}), x_{2i_A}(p_{12})) \\ s_{1i_A}(p_{12}) &= \bar{\omega}_{1i_A} - x_{1i_A}(p_{12}) \\ d_{2i_A}(p_{12}) &= x_{2i_A}(p_{12}) \end{aligned}$$

for $i_A = 1, ..., I_A$, and

$$\begin{aligned} x_{i_B}(p_{12}) &= (x_{1i_B}(p_{12}), x_{2i_B}(p_{12})) \\ d_{1i_B}(p_{12}) &= x_{1i_B}(p_{12}) \\ s_{2i_B}(p_{12}) &= \bar{\omega}_{2i_B} - x_{2i_B}(p_{12}) \end{aligned}$$

for $i_B = I_A + 1, ..., I$.

Then, by aggregating over the traders, one obtains the aggregate gross demand functions, as well as the aggregate net demand and supply functions for either commodity, that is:

$$\begin{aligned} x_1(p_{12}) &= \Sigma_{i=1}^{I} x_{1i}(p_{12}) = \Sigma_{i_A=1}^{I_A} x_{1i_A}(p_{12}) + \Sigma_{i_B=I_A+1}^{I} x_{1i_B}(p_{12}), \\ d_1(p_{12}) &= \Sigma_{i_B=I_A+1}^{I} d_{1i_B}(p_{12}) = \Sigma_{i_B=I_A+1}^{I} x_{1i_B}(p_{12}), \\ s_1(p_{12}) &= \Sigma_{i_A=1}^{I_A} s_{1i_A}(p_{12}) = \Sigma_{i_A=1}^{I_A}(\bar{\omega}_{1i_A} - x_{1i_A}(p_{12})) = \bar{\omega}_1 - \Sigma_{i=1}^{I_A} x_{1i_A}(p_{12}), \end{aligned}$$

and

$$\begin{aligned} x_2(p_{12}) &= \Sigma_{i=1}^{I} x_{2i}(p_{12}) = \Sigma_{i_A=1}^{I_A} x_{2i_A}(p_{12}) + \Sigma_{i_B=I_A+1}^{I} x_{2i_B}(p_{12}), \\ d_2(p_{12}) &= \Sigma_{i_A=1}^{I_A} d_{2i_A}(p_{12}) = \Sigma_{i_A=1}^{I_A} x_{2i_A}(p_{12}), \\ s_2(p_{12}) &= \Sigma_{i_B=I_A+1}^{I} s_{2i_B}(p_{12}) = \Sigma_{i_B=I_A+1}^{I}(\bar{\omega}_{2i_B} - x_{2i_B}(p_{12})) = \bar{\omega}_2 - \Sigma_{i_B=I_A+1}^{I} x_{2i_B}(p_{12}), \end{aligned}$$

from which one can derive the aggregate excess demand functions:

$$\begin{aligned} z_1(p_{12}) &= d_1(p_{12}) - s_1(p_{12}) = \Sigma_{i=I_A+1}^{I} x_{1i}(p_{12}) - \bar{\omega}_1 + \Sigma_{i=1}^{I_A} x_{1i}(p_{12}) = \\ &= \Sigma_{i=1}^{I} x_{1i}(p_{12}) - \bar{\omega}_1 = x_1(p_{12}) - \bar{\omega}_1, \\ z_2(p_{12}) &= d_2(p_{12}) - s_2(p_{12}) = \Sigma_{i=1}^{I_A} x_{2i}(p_{12}) - \bar{\omega}_2 + \Sigma_{i=I_A+1}^{I} x_{2i}(p_{12}) = \\ &= \Sigma_{i=1}^{I} x_{2i}(p_{12}) - \bar{\omega}_2 = x_2(p_{12}) - \bar{\omega}_2. \end{aligned}$$

Finally, by setting the aggregate excess demand functions equal to zero, one obtains the market clearing equations, one for each commodity:

$$z_l(p_{12}^W) = x_l(p_{12}^W) - \bar{\omega}_l = 0, \qquad l = 1, 2. \tag{8}$$

Yet, given Walras' Law, that is, $p \cdot z = p_{12} z_1(p_{12}) + z_2(p_{12}) \equiv 0$, $\forall p = (p_{12}, 1) \in \mathbb{R}_{++} \times \{1\}$, only one equation provides an independent equilibrium condition. By solving either equation, therefore, we obtain the Walrasian equilibrium relative price, p_{12}^W. The Walrasian equilibrium allocation, $\left(x_i^W\right)_{i=1}^I = \left(x_{1i}^W, x_{2i}^W\right)_{i=1}^I$, can then be obtained by plugging the Walrasian equilibrium price into the individual gross demand functions, that is, $\left(x_i^W\right)_{i=1}^I = \left(x_i\left(p_{12}^W\right)\right)_{i=1}^I$ (Walras 1988, pp. 136-7). Under the stated conditions, a "solution" exists, even if it is not necessarily unique (Walras 1988, p. 97). Yet, for simplicity, we shall assume uniqueness in the following.

By setting $I_A = I_B = 1$, hence $I = 2$, one obtains a Walrasian model of a pure-exchange, two-commodity, two-trader economy, which can be represented by means of an Edgeworth Box diagram (see Fig. 2 below). By reverting to the notation $i = A, B$, let $x^W = \left(x_A^W, x_B^W\right) = \left(\left(x_{1A}^W, x_{2A}^W\right), \left(x_{1B}^W, x_{2B}^W\right)\right)$ denote the Walrasian equilibrium allocation in Fig. 2. The ray from ω through x^W is the equilibrium budget line for either trader, the absolute value of its slope ($\tan \alpha^W$) representing the Walrasian equilibrium price p_{12}^W.

Fig. 2 about here

The method of "solution" suggested above is "analytique", for it consists in finding the roots of a system of ordinary algebraic equations. But one can also restate the problem in geometrical terms, that is, one can give it "la forme géométrique", by drawing for each commodity the demand and supply curves and finding their intersection. Alternatively, when there are just two cornered traders in the economy, one can plot the graphs of their respective gross demand functions, $x_A(p_{12})$ and $x_B(p_{12})$, in the Edgeworth Box diagram; then, finding the intersection of the two graphs, called "offer curves" after Johnson (1913, p. 487) and labelled o_A and o_B in Fig. 2 above, provides another geometrical method for identifying the Walrasian equilibrium allocation and, implicitly, the Walrasian equilibrium price, too. Anyhow, the "solution" thus determined, by means that can indifferently be "analytique" or "géométrique", still remains, for Walras (1988, p. 137), "la solution mathématique". Then one can ask how such "solution" is concretely determined "sur le marché".

Since the distinction between "mathematical" (or "theoretical", or "scientific") and "empirical" (or "practical") "solution" is of fundamental importance in Walras's system of thought, it is worth quoting in full a passage from the 1874 *mémoire*, where he tries to clarify the meaning of the two "solution" concepts:

> *A priori*, ce problème est évidemment soluble, du moins en principe, par le procédé mathématique, comme il est soluble, en fait, sur le marché, par le procédé empirique de la hausse et de la baisse. Sur notre marché, nous avons supposé les acheteurs et les vendeurs en présence les uns des autres ; mais la présence de ces échangeurs n'est pas nécessaire : qu'ils donnent leurs ordres à des agents, le marché se tiendra entre ces derniers. [...] Mais, théoriquement, la présence des agents est-elle plus nécessaire que celle des échangeurs eux-mêmes? Pas le moins du monde. Ces agents sont les exécuteurs purs et simples d'ordres inscrits sur des carnets : qu'au lieu de faire la criée, ils donnent ces carnets à un calculateur, et ce calculateur déterminera les prix d'équilibre non pas certes aussi rapidement, mais à coup sûr plus rigoureusement que cela ne pourrait se faire par le mécanisme de la hausse et de la baisse. Nous sommes ce calculateur ; nos courbes de demande répresentent les ordres des échangeurs [...]. (Walras 1874, p. 37)

As can be seen, to find "la solution mathematique" of the exchange problem means precisely, for Walras, to compute the equilibrium values of the relevant variables, what could be done, at least in principle, by a "calculateur" endowed with the required information, which includes the traders' demand functions. But since this sort of information is hardly ever fully available, "la solution théorique serait, dans presque tous les cas, absolument impraticable" (Walras 1988, p. 93). Hence, it is the "empirical solution" provided by the market which must be relied upon in order to find out the equilibrium values of the relevant variables.

According to Walras, this empirical determination occurs as follows. One relative price p'_{12} being "crié", the corresponding excess demand for commodity 1, $z_1(p'_{12})$, is determined "sans calcul, mais néanmoins conformément à la condition de satisfaction maxima"; then p'_{12} increases or decreases, according to whether $z_1(p'_{12})$ is greater or less than zero, and the process, which is just one manifestation of the time-honoured "loi de l'offre et de la demande", goes on until the equilibrium price, p^W_{12}, is eventually reached (Walras 1988, pp. 137-8). Once again, when there are just two traders in the economy, the working of the process can be graphically illustrated in the Edgeworth Box diagram (see Fig. 2, where the process is suggested by the clockwise rotation of the budget line from the initial position, where the absolute value of the slope is $p'_{12} = \tan\alpha'$, towards the final equilibrium position). This is the first and simplest instance of Walras's *tâtonnement* construct, a construct that Walras will then systematically apply to all his equilibrium models with a view to explaining how, in each case, "la solution mathématique" is empirically attained "sur le marché" (see Walras 1876a, p. 69; 1988, pp. 173, 189).

With his theory of *tâtonnement*, Walras is apparently able to provide an answer to the issue of equilibrium attainment, an issue that, as explained in the previous Section, Jevons had deliberately set aside. Yet Walras's approach is far from unobjectionable. As a matter of fact, in the early 1870s, when Walras was starting to develop his formal analysis of the equilibration process in the exchange model, he probably believed the dynamics of the *tâtonnement* process not to be incompatible with some sort of observable disequilibrium behaviour, even if such behaviour is nowhere formalised in his writings of that period[12]. Yet, after the appearance of Bertrand's critical review of Walras's *Théorie mathématique de la richesse sociale* (1883), where the point was made that any sort of observable disequilibrium trading would necessarily entail some change in the data of the theory, hence in the equilibrium eventually reached by the equilibration process, Walras was led to explicitly introduce a 'no trade out of equilibrium assumption', thereby turning the *tâtonnement* process in the exchange model into a purely virtual process in 'logical' time, over which nothing observable is allowed to take place[13]. So, in the end, not differently from Jevons's "state of equilibrium", also Walras's equilibrium price and allocation must be supposed to be "instantaneously" arrived at, as Walras himself explicitly acknowledged (1885, p. 312, fn. 1), even if in Walras's case the attainment of equilibrium is apparently supported by a 'durationless process', i.e., a process consuming no amount of 'real' time to carry its effects through.

Let us now turn to the Law of One Price. As can be seen from the above discussion of Walras's pure-exchange, two-commodity model, the role of such Law is pervasive in the Walrasian theoretical system: not only does the Law of One Price play a central role in the definition of Walras's equilibrium concept, but it also holds an irreplaceable position in the Walrasian analysis of the equilibration process. As far as the definition of equilibrium is concerned, since his first *mémoire* Walras is very clear about the requirement that the equilibrium price must be unique in every market:

> Les prix résultent mathématiquement de courbe de domande en raison de ce fait qu'il ne doit avoir, sur le marché, qu'un seul prix, celui pour lequel la demande totale effective est égale àl'offre totale effective [...]. (Walras 1874a, p. 43-4; see also 1988, pp. 142, 200)

[12] Walras's original belief that the *tâtonnement* construct may not be inconsistent with the traders carrying out actual observable trades when markets are out of equilibrium can be indirectly inferred, e.g., from the introductory examples of the disequilibrium working of specific markets (the market for corn and the market for consols), respectively developed for illustration purposes in Walras's first *mémoire* (1874a, pp. 31-2) and in the first edition of *EEPP* (1988, pp. 71-2, **1**).

[13] See Walras (1885, p. 312, fn. 1, and 1988, pp. 71-2, **2-5**); see also the *mémoire* attached to a letter sent by Walras to Pareto, as reproduced in Jaffé (1965, Vol. II, p. 630). In the fourth edition of *EEPP*, appearing in 1900, Walras, by eventually adopting the so-called "hypothèse des *bons*", was able to generalize the 'no trade out of equilibrium assumption' to all sorts of models and *tâtonnement* processes (1988, pp. 309, 377, 447, **4-5**). On Walras's *tâtonnement* construct and equilibrium concept, see Donzelli (2007).

As far as the analysis of the equilibration process the *tâtonnement* process in the pure-exchange, two-commodity model could not even start, as we have just seen, if a unique relative price were not quoted, and could not continue if that price were not progressively adjusted according to the law of supply and demand.

In Walras's pure-exchange model, the ubiquitous presence of the Law of One Price in both equilibrium and equilibration analysis is certainly not accidental. The reason for this is simple: the Walrasian excess-demand apparatus rests on individual price-taking behaviour, which in turn depends on the parametric role of prices, hence on the Law of One Price; but in Walras's pure-exchange model the excess-demand apparatus is crucially employed in both the characterisation of the Walrasian equilibrium concept (for the market equilibrium conditions are provided by the market-clearing equations, where the excess-demand functions are set equal to zero) and the shaping of the *tâtonnement* process (for the rates of change of prices are sign-preserving functions of the excess demands in the respective markets); therefore, the Law of One Price is the foundation on which the whole of Walras's theoretical structure rests.

This raises the question of the logical status of such Law in Walrasian theory: Is it assumed as a postulate or is it derived as a theorem from still more basic principles? Not differently from Jevons, probably also Walras would have desidered to make the Law emerge as the result of the interaction of self-interested, fully informed individuals in well-organised, competitive markets. The following passage seems to point in that direction:

> Les marchés les miex organisés sous le rapport de la concurrence sont ceux où les ventes et achats se font à la criée, par l'intermédiaire d'agents tes qu'agents de change, courtiers de commerce, crieurs, qui les centralisent, de tel sorte qu'aucun échange n'ait lieu sans que les conditions en soient annoncées et connues et sans que les vendeurs puissent aller au rabais et les achéteurs à l'enchère. (Walras 1988, p. 70)

The last words in the above quote seem to foreshadow a theory of arbitrage, which might help explain or at least corroborate the emergence of uniform prices in well-organised markets. As a matter of fact, Walras does make use in *EEPP* (1988, pp. 161-173, **1** and **2**-**4**) of a theory of arbitrage, essentially inherited from Cournot (1838, pp. 29-43), to explain the formation of a consistent 'price system' in a pure-exchange economy with an arbitrary finite number of commodities, greater than two. Yet, Walras's argument is defective and the scope of this particular application of Cournot's theory of arbitrage is limited. On the other hand, the easily identifiable shortcomings of this specific endeavour help understand why no other attempt is made by Walras, in either *EEPP* or any of his writings, to support the Law of One Price by means of Cournot's or, for that matter, any other theory of arbitrage.

Let us see how Walras actually employs Cournot's theory in *EEPP*. In passing from the model of a pure-exchange, two-commodity economy to the model of an economy with several commodities (say, m commodities, with $m > 2$), a new problem arises: for the notion of a 'relative price', characteristic of the two-commodity model, does not immediately generalise to the notion of a 'price system', appropriate to the m-commodity model ($m > 2$). Walras perceives that the latter notion is, in a sense, an equilibrium concept, which ought to be theoretically derived from more primitive concepts and axioms by means of some sort of equilibration process. In the light of this, in developing his m-commodity model, Walras does not assume from the start the existence of a 'price system', that is, the existence of an m-dimensional price vector, but he tries to build it up, precisely by means of a theory of arbitrage inspired by Cournot. To this end, he assumes the existence of $\frac{m(m-1)}{2}$ "special markets", where as many pairs of commodities are traded under the condition that reselling is forbidden; next, by exploiting the results already obtained in the pure-exchange, two-commodity model, he supposes $\frac{m(m-1)}{2}$ "imperfect equilibrium" relative prices between pairs of commodities to be established on the corresponding "special markets"; then, by relaxing the 'no reselling' assumption, he shows that 1) the demand conditions would change due to the appearance of new "supplementary" demands for the purpose of indirect trading, 2) the "imperfect equilibrium" relative prices initially arrived at would be disrupted, and finally 3) such disruption would persist until a "perfect" or "general equilibrium" 'price system', satisfying Cournot's arbitrage conditions, were to be established.

Now, Walras's analysis of the emergence of a consistent m-dimensional 'price system' from $\frac{m(m-1)}{2}$ possibly inconsistent relative prices is unsatisfactory for at least two reasons: first, in spite of the 'no reselling' and other limiting assumptions, the 'two-commodity' model proves insufficient for allowing the determination of the "imperfect equilibrium" relative prices in the "special markets", which are therefore not explained by Walras, but simply assumed to start with; secondly, the standard Walrasian theory of demand for consumer goods proves inadequate for explaining the "supplementary" demands for the purpose of indirect trading which originate from the relaxation of the 'no reselling' assumption. As can be seen, the fundamental reason behind the inadequacy of Walras's explanation of the emergence of a consistent 'price system' in an m-commodity model is that he tries to explain new phenomena (equilibrium conditions in a multi-commodity world, "supplementary" demand for the purpose of indirect trading) by using the tools he has already at hand (standard equilibrium theory in a two-commodity world, standard theory of demand for consumer goods); but such tools are appropriate for the old phenomena only.

As suggested above, this conclusion helps explain why Walras does not even try to use Cournot's or any other theory of arbitrage to derive the Law of One Price from more basic principles. The fact is that, in order to pursue this aim, one ought to be able to explain what happens when the Law does not hold, hence uniform prices do not yet exist. In order to accomplish this task, however, one would need to rely on a theory of individual behaviour that does not assume price uniformity. But the only theory of individual behaviour on which Walras can rely is a theory of price-taking behavior, which presupposes the Law of One Price to be already in operation and effective; hence the Law of One Price cannot be explained, within Walras's theoretical system, but must be taken as an axiom[14].

4 Edgeworth on Jevons and Walras

The point departure for Edgeworth's discussion of the exchange problem in *MP* is precisely represented by Jevons's "theory of exchange", as developed in Chapter 4 of *TPE* (Edgeworth 1881, pp. 20, 39, and App. V, pp. 104-16). Yet, instead of accepting Jevons's basic idea that an exchange model involving just two "trading bodies" is sufficient to cover the whole range of *a priori* possible trade situations, Edgeworth clearly distinguishes from the very start two separate theories: the "theory of simple contract", which concerns just two individual traders, and the "theory of exchange" proper, which deals instead with a greater number of traders, more or less competing with each other (1881, pp. 29, 31, 109). Edgeworth's fundamental conjecture is that the intensity of competition among the traders increases with their number; moreover, the greater the competition among the traders, the more determinate is the outcome of the trading process; as a consequence, "perfect competition" and "perfectly determinate" outcome can only obtain when the traders' number is "practically infinite". In between the two extremes of an "isolated couple" of traders, on the one hand, and a "perfect" market populated by a "practically infinite" number of traders, on the other, there is of course a whole range of intermediate situations (1881, pp. 20, 39, 146-7).

So, right at the beginning of *MP*, Edgeworth raises his fundamental question: "How far contract is indeterminate". The "general answer" he is ready to offer is as follows:

> (α) Contract without competition is indeterminate, (β) Contract with perfect competition is perfectly determinate, (γ) Contract with more or less perfect competition is less or more indeterminate. (Edgeworth, 1881, p. 20)

In developing his "theory of simple contract" Edgeworth employs the very same model of a two-commodity economy with two cornered traders as the one used by Jevons. Hence, Edgeworth's pure-exchange, two-trader, two-commodity economy, denoted by $\mathcal{E}_E^{2\times 2} = \left\{ \left(\mathbb{R}_+^2, u_i\left(\cdot\right), \omega_i \right)_{i=A,B} \right\}$, being the same as Jevons's, can be described as before by means of an Edgeworth Box diagram[15]. Unlike

[14] It should be stressed that the Law is postulated by Walras also when, as we have seen, he actually makes some use of Cournot's theory of arbitrage to provide a limited explanation of the emergence of a consistent 'price system' in a multi-commodity economy: for, even then, he postulates the existence of uniform prices in all the "special markets" for pairwise exchanges of commodities.

[15] The sbscript E stands for Edgeworthian. The diagrams of the pure-exchange, two-trader, two-commodity economy actually appearing in *MP* (1881, pp. 28, 114) are drawn in the four-

Jevons, Edgeworth (1881, pp. 21) precisely defines the "contract curve", which, as before, is set of the allocations satifying equations (2) above, that is, the set of Pareto-optimal allocations. Edgeworth (pp. 19, 29) also identifies the "set of the final settlements", which is the subset of the "contract curve" contained in the lens-shaped region of the plane enclosed between the traders' indifference curves passing through the endowment allocation; it is the set of the Pareto-optimal allocations, $x^C = \left(x_A^C, x_B^C\right)$, which also satisfy the individual rationality conditions, $u_i(x_i) \geq u_i(\omega_i)$, $i = A, B$. Edgeworth's "set of the final settlements" corresponds to what, in the current terminology of coalitional game theory, would be called the "core" of the Edgeworth Box economy. In Fig. 3 below, the "contract curve" is the locus connecting the origins O_A and O_B, while the "set of the final settlements" is the portion of the contract curve going from x^{1r} to x^{1l}, where x^{1r} and x^{1l} respectively are the right and left intersection points between the contract curve and the traders' indifference curves through the endowment allocation.

Figure 3 about here

According to Edgeworth, in the model of an Edgeworth Box economy, the "[final] settlements [...] are represented by an indefinite number of points" (Edgeworth, 1881, p. 29). Such "indeterminateness" is the hallmark of the "theory of simple contract". The reasons underlying such "indeterminateness" can be easily spelled out. The two traders will trade if and only if their "acts of exchange" are (weakly) mutually advantageous, that is, if they increase the utility of at least one of them, without decreasing the utility of either trader; in the wake of Feldman (1973, p. 465), let us call such "acts", which may be 'finite' or 'differential', "bilateral trade moves". Now, if $MRS_{21}^A(\omega_A) \neq MRS_{21}^B(\omega_B)$, there are infinitely many bilateral trade moves that can be carried out by the traders at the start of the bargaining process; a similar situation, however, would arise over and over again at each allocation that might be reached after any move is effected, as long as the traders' marginal rates of substitution diverge.

In view of this, lacking any further detailed information about the traders' knowledge, attitudes, skills, etc., the actual path followed by the two traders during the bargaining process cannot be specified: for it depends on idiosyncratic circumstances which are specific to either trader and, as such, irreducible to general rules or systematic theorizing (1881, pp. 29-30). What can be stated for sure, under the trading rule specified above, is that 1) the process will come to an end (perhaps in the limit), and 2) its end point, that is, the "final settlement" reached through the bargaining process, will be such that the traders' marginal rates of substitution are the same. Yet, if the path cannot be specified, its end point cannot be identified either: the path will certainly end up some-

coordinate space of the traders' net trades, rather than in the four-coordinate space of the traders' allocations, as is common nowadays, in the wake of Pareto's introduction of the latter kind of diagram in 1906. In this paper we have preferred to keep to the currently prevailing representation. In any case, the choice of which representation to adopt is immaterial, since there exists a one-to-one correspondence between the two.

where in the core of the Edgeworth Box economy, but which specific allocation is eventually reached it is impossible to say.

By using a notation similar to that introduced in the previous Section, let us now provide an "illustration" of a possible path from the initial endowment allocation to a "final settlement" in the core of $\mathcal{E}_E^{2\times 2} = \left\{ \left(\mathbb{R}_+^2, u_i(\cdot), \omega_i \right)_{i=A,B} \right\}$. It is no more than an "illustration" for, according to Edgeworth,

> only the position of equilibrium is knowable, not the path by which the equilibrium is reached. As Jevons says, "It is a far more easy task to lay down the conditions under which trade is completed and interchange ceases, than to attempt to ascertain at what rate trade will go on when equilibrium is not attained." Particular paths may be indicated by way of illustration, "to fix the ideas", as mathematicians say. (Edgeworth, 1904, pp. 39-40)

An identical stance, concerning the impossibility of any "general" dynamical deterministic theory of the equilibration process, will be taken by Edgeworth over and over again, from beginning to end of his long scientific life. As the following quotes will make clear, Edgeworth's 'dynamical impossibility conjecture', as it might be called, is traced back by Edgeworth himself to Jevons's original position, already discussed in Section 2 above, about statics, dynamics, equilibrium, and equilibration. Moreover, according to Edgeworth, his own 'dynamical impossibility conjecture' does not only concern the "theory of simple contract", which is presently under discussion, but extends also to the "theory of exchange" proper, that will be taken up shortly.

The first quote is drawn from Edgeworth's scathing review of the second edition of Walras's *EEPP* (1889), which will start the Edgeworth-Bortkiewicz controversy chiefly focusing on Walras's theory of *tâtonnement*[16]:

> Now, as Jevons points out, the equations of exchange are of a statical, not a dynamical, character. They define a position of equilibrium, but they afford no information as to the path by which that point is reached. (Edgeworth 1889a, p. 268)

This passage is reproduced, in French translation, in Edgeworth's reply to Bortkiewicz (1891a, p. 364). In the same article, shortly after that self-quotation, in discussing the issue of equilibrium determination, or equilibrium establishment, in a Walrasian competitive market, Edgeworth writes:

> Les forces en jeu dans le système, si l'on peut ainsi parler, étant données, la position d'équilibre vers laquelle tend tout le système se trouve par là même déterminé. Mais je maintiens que le jeu de tout ce marchandage par lequel le prix du marché se trouve déterminé, la direction que suit le système pour arriver à la position d'équilibre, ne rentre pas dans la sphère de la science. (Edgeworth, 1891a, p. 364)

[16] On this, see Donzelli (2009).

In the same paper, about a page after the preceding passage, Edgeworth goes back to the analogy established by Jevons between the theory of exchange and the theory of the lever and its implications for economic analysis:

> [Jevons] a soin de nous faire remarquer que sa théorie est analogue à la théorie du levier (*The Theory*, 2e édit., pp. 110-114).
>
> Sa théorie se réfère seulement "à la vitesse virtuelle" et non à un mouvement véritable. Ses équations expriment une position d'équilibre économique, mais elles ne nous fournissent aucune information sur le jeu de l'offre et de la demande par lequel cette position se trouve atteinte. (Edgeworth 1891a, pp. 365-6)

Finally, in one of the his last writings (a "Note" appended to the reprint of the 1889 Presidential Address in Volume II of the *Papers Relating to Political Economy*, published in 1925), Edgeworth returns once again to his 'dynamical impossibility conjecture':

> [We] have no general *dynamical* theory determining the path of the economic system from any point assigned at random to a position of equilibrium. We know only the statical properties of the position; as Jevons's analogy of the lever implies (*Theory*, p. 110 *et seq.*). (Edgeworth, 1925, p.311)

This being said, let us go back to our "illustration" (see Fig. 3 above). In order to "fix the ideas", let us suppose that the bilateral trade moves are 'finite', so that a "final settlement" is reached after a finite sequence of moves, say after $\Theta \in \mathbb{N}$ moves, where $\Theta > 1$. Let $t_0 \in \mathbb{Z}$ denote the trade round (a finite time interval) over which the trading process is supposed to take place. Each move is indexed by $\theta = 1, ..., \Theta$. The first move takes the economy from the endowment allocation $x_{t_0}^0 = \omega_{t_0}$ to the allocation $x_{t_0}^1$; the generic move θ carries the economy from $x_{t_0}^{\theta-1}$ to $x_{t_0}^{\theta}$; the last move takes the economy from $x_{t_0}^{\Theta-1}$ to $x_{t_0}^{\Theta}$. The broken line connecting the successive allocations in the finite sequence $\left(x_{t_0}^{\theta}\right)_{\theta=0}^{\Theta}$ describes the path followed by the traders during the trading process in the Edgeworth Box. The absolute value of the slope of the straight line segment connecting any two successive allocations in the sequence $\left(x_{t_0}^{\theta}\right)_{\theta=0}^{\Theta}$ defines the rate of exchange implicit in the move leading to the allocation located at the end of the segment: so, for the generic pair of allocations $x_{t_0}^{\theta-1}$ and $x_{t_0}^{\theta}$, the rate of exchange implicit in the θ-th move is given by $\frac{\Delta x_{2t_0}^{\theta}}{\Delta x_{1t_0}^{\theta}} = \left| \frac{x_{2At_0}^{\theta} - x_{2At_0}^{\theta-1}}{x_{1At_0}^{\theta} - x_{1At_0}^{\theta-1}} \right| = \left| \frac{x_{2Bt_0}^{\theta} - x_{2Bt_0}^{\theta-1}}{x_{1Bt_0}^{\theta} - x_{1Bt_0}^{\theta-1}} \right| \geq 0$. Similar results would obtain in case the bilateral trade moves were 'infinitesimal', rather than 'finite': in that case, assuming the path followed during the trading process to be described by a smooth curve with a continuously changing slope, everywhere well-defined, then the infinitesimal rate of exchange $\frac{dx_{2t_0}^{\theta}}{dx_{1t_0}^{\theta}}$ ruling at each allocation $x_{t_0}^{\theta}$ reached during the process would coincide with the absolute value of the slope of the curve at that point (Edgeworth 1881, pp. 39, 105).

According to Edgeworth, the rate of exchange implicit in any trade move can, and generally does, vary along the path, in a way that is largely unpredictable. The only property of the rate of exchange which can be precisely predicted is that the rate of exchange implicit in the last move Θ, that is, in the move leading to the "final settlement" $x_{t_0}^{\Theta}$, must equal the common value of the marginal rates of substitution of the two traders evaluated at that point: namely, if the last move is 'finite' (resp., 'infinitesimal'), one must have $\frac{\Delta x_{2t_0}^{\Theta}}{\Delta x_{1t_0}^{\Theta}}$ (resp., $\frac{dx_{2t_0}^{\Theta}}{dx_{1t_0}^{\Theta}}$) $= MRS_{21t_0}^{A}(x_{t_0}^{\Theta}) = MRS_{21t_0}^{B}(x_{t_0}^{\Theta})$ (Edgeworth 1881, pp. 21-3, 109).

As recalled above, the model employed by Edgeworth in developing his "theory of simple contract" is formally the same as Jevons's. Yet, the variability of the rate of exchange over the trading process, assumed to be the typical occurrence in Edgeworth's "theory of simple contract", stands out against the uniformity of the rate of exchange, postulated by Jevons in his model. This "contrast", which also bears on the interpretation and use of the Law of Indifference, is explicitly recognised by Edgeworth (1881, p. 108), who devotes an entire Appendix of *MP* (Appendix V, pp. 104-116) to the elucidation of the differences and similarities existing between his approach and Jevons's. The following passage is particularly revealing:

> It has been prominently put forward in these pages that the Jevonian 'Law of Indifference' has place only where there is competition, and, indeed, *perfect* competition. Why, indeed, should an isolated couple exchange every portion of their respective commodities at the same rate of exchange? Or what meaning can be attached to such a law in their case? The dealing of an isolated couple would be regulated not by the theory of exchange (stated p. 31), but by the theory of simple contract (stated p. 29).
>
> This consideration has not been brought so prominently forward in Professor Jevons's theory of exchange, but it does not seem to be lost sight of. His couple of dealers are, I take it, a sort of typical couple, clothed with the property of 'Indifference' [...]; an individual dealer only is presented, but there is presupposed a class of competitors in the background. (Edgeworth 1881, p. 109)

As can be seen, a literal interpretation of Jevons's model, which should view it as describing the bargaining activities of two isolated traders, would lead, according to Edgeworth, to results totally at variance with those obtained by Jevons himself. But Jevons's results, including his Law of Indifference, can be saved, provided that his model is not taken literally (as has been done, with some justification, by a number of critics), but reinterpreted as a model of a perfectly competitive market, populated by infinitely many traders. This is what Edgeworth sets out to do, by combining his recontracting mechanism, which ensures convergence of the trading process to the core of an economy with a fixed number of traders, with his replication mechanism, which, by increasing the size of the economy, ensures convergence of the trading process to a perfectly competitive equilibrium of the Jevonsian or Walrasian type. Yet, before examining these developments concerning Edgeworth's "theory of exchange" proper, it is still necessary to discuss nature and timing of the bargaining process assumed in his "theory of simple contract".

In *MP*, nature and timing of the trading process are not discussed in any detail, as Edgeworth himself will acknowledge ten years later in his reply to Bortkiewicz, at least as far as the "time" factor is concerned (1891a, p. 367). Yet, at least as regards the "theory of simple contract", Edgeworth appears to conceive the trading process as taking place over the trade round through a sequence of observable, irreversible, piecemeal trades, effectively carried out by the traders one after the other at different rates of exchange: the first paragraph in the last quoted passage, as well as other hints scattered all along *MP* (1881, pp. 24-5, 28-30, 115-6), seem to confirm that this is Edgeworth's original interpretation of the trading process in the case of two traders.

Such belief is indisputably corroborated by the first part of a paper in Italian, published in 1891 (Edgeworth 1891b), where the author critically discusses Marshall's theory of barter, as stated in a "Note on Barter" appearing in the first edition of Marshall's *Principles of Economics*, published the year before[17]. In his paper Edgeworth starts by acknowledging that

> Marshall [...] has avoided the common error of attributing to two persons who are bargaining with each other a fixed rate of exchange governing the whole transaction. A uniform rate of exchange, he remarks, is applicable only to the case of perfect market. (Edgeworth, 1925d, p. 315; English "translation" of 1891b, p. 234)

He then proceeds to discuss an illustration of the barter process put forward by Marshall in his "Note", concerning two traders, A and B, trading apples for nuts. With reference to Marshall's illustration, Edgeworth draws a picture (Edgeworth, 1925d, p. 316, Fig. 1; 1891b, p. 236, Fig. 1), where an Edgeworth Box diagram is plotted in the space of net trades; in that diagram there appears a "series of short lines", that is, a "broken line", exactly corresponding to the broken line drawn in Fig 3 above. Commenting upon the meaning of such drawing, Edgeworth claims that it "corresponds to successive barters (at different rates of exchange) of a few nuts for a few apples" (1925d, p. 316; 1891b, p. 236). Now, as the last sentence conclusively shows, Edgeworth, at least when referring to Marshall's illustration, conceives of the trading process as a sequence of observable, irreversible, piecemeal exchanges, taking place at successive time instants distributed over a given trade round. For further reference, let us label

[17] A partial English "translation" of the 1891b paper in Italian appears in Volume II of Edgeworth's *Papers Relating to Political Economy* (see 1925d). Edgeworth's 1891 paper gives rise to a lively controversy, involving, besides Edgeworth himself (1891c), also Marshall (see 1961a, pp. 791-3, and 1961b, pp. 791-8), and, at Marshall's instigation, the young mathematician Berry (1891). On this issue, see Donzelli (2009, pp. 32-6).

this first interpretation of Edgeworth's trading process in the "theory of simple contract" as the 'Marshallian' one.

Yet, quite unexpectedly, in a footnote appended to a passage by Marshall concerning the potential multiplicity of end points (also referred to as "equilibria") in the barter process concerned, Edgeworth offers a significantly different interpretation of both the time structure and the very nature of the process under question:

> It should be noted that such a multiplicity of equilibria occurs if we suppose that A and B, without actually trading successive doses of apples and nuts, and therefore without actually experiencing either any shortage or plenty of the traded goods, only think of such trades, or discuss them, while bargaining with one another, and mentally appreciate their hedonistic effects. (Edgeworth, 1891b, p. 235, fn. 1; our translation[18])

As far as the "theory of simple contract" is concerned, the purely mentalistic interpretation of the trading process offered in the second part of Edgeworth's 1891 paper, which unexpectedly reverses the 'Marshallian' interpretation endorsed just a few lines before, is a relative novelty in Edgeworth's thought: for, as suggested above, the 'Marshallian' interpretation had apparently dominated the scene in *MP*. Yet, some sort of mentalistic interpretation of the equilibration process is certainly not an absolute novelty in Edgeworth's reflections: for, as far as the "theory of exchange" proper is concerned, a mentalistic interpretation of the recontracting process had already been put forward in *MP*, as we shall see in greater detail later in this Section. For further reference, let us call 'mentalistic' the second interpretation of Edgeworth's trading process in the "theory of simple contract", as proposed in the second part of the 1891b paper.

Such 'mentalistic' interpretation of Edgeworth's process, closely resembling such practices as those referred to as 'cheap talk' in contemporary game theory, rules out all observable or irrevocable behaviour until a core allocation is eventually reached. In this respect, it also reminds one of some features of Walrasian *tâtonnement*, when viewed as a virtual process in 'logical' time. Yet, the implications of the 'mentalistic' interpretation of an Edgeworthian process are even stronger than those entailed by a virtual interpretation of a Walrasian *tâtonnement*: in fact, during a virtual *tâtonnement*, no observable trades can be carried out, but prices are supposed to be quoted all the same, so that the process can be described, both analytically and geometrically, as suggested, e.g., in Fig. 2 above; on the contrary, during a 'mentalistic' Edgeworthian process, not only no observable trades are carried out, but also no rates of exchange are supposed to be openly quoted by anybody, so that no analytical or geometrical description of the process, of the type suggested in Fig. 3 for the 'Marshallian'

[18] This footnote belongs to a part of the 1891 paper in Italian (1891b) that is not reproduced in the later abridged English "translation" (1925d). Edgeworth will reaffirm the position expressed in this footnote a few months later, in his rejoinder (1891c, p. 317) to Berry's reply (1891) to Edgeworth's original paper in Italian (1891b).

interpretation, ought to be provided. As a matter of fact, Edgeworth is silent on this issue, but his insistence on the impossibility of any "general" dynamical deterministic theory of the equilibration process would seem to militate against any representation of the trading process, when 'mentalistically' interpreted, even for purely illustrative purposes: under the 'mentalistic' interpretation, therefore, no path should be drawn in Fig. 3, but only the end point of the process, say x_t^M, where M stands for 'mentalistic', might be plotted therein.

This is not the end of the story, however. For Berry, both in his reply to Edgeworth's 1891b paper in Italian (Berry 1891, p. 552) and in a private letter addressed to Edgeworth (published in Marshall 1961b, p. 794), unearths yet another possible interpretation of Edgeworth's trading process which, though probably already implicit in some occasional real-world examples of bargaining between "landlords and tenants" or "workmen and employers" scattered through *MP* (1881, pp. 17, 134-48), had not received due attention up until the 1891 controversy. This third interpretation, which might be called 'stationary', will be explicitly endorsed by Edgeworth only after a long period of reflection (1904, p. 40), becoming relatively more popular afterwards (1907, pp. 526 ff., and 1925c, p. 313): it differs from the other two interpretations for it supposes the trading process to take place over a sequence of trade rounds, rather than over one single trade-round, under unchanging economic conditions (whence the label 'stationary'). In the 'stationary' interpretation the trades are observable and irrevocable, as in the 'Marshallian' interpretation, but contrary to what is assumed in the 'mentalistic' one; yet, the effects of such trades are provisional, as in the 'mentalistic' interpretation, but contrary to what is supposed in the 'Marshallian' one: this is due to the assumed impermanence of both the stipulated contracts, whose validity is limited to the trade round in which they are signed, and their consequences, which do not outlast the period of validity of the contracts from which they ensue, that is, the same trade round in which the contracts are signed.

Edgeworth's idea, as can be gathered from his sparse remarks, is that the trading process, through repeated interactions between the traders under stationary conditions (namely, constant consumption sets, endowments, and utility functions), progressively converges by trial and error to a core allocation, which however remains as indeterminate in this case as it was under the alternative interpretations. As with the 'Marshallian' interpretation, also with the 'stationary' one, one can depict the path followed by the trading process in an Edgeworth Box diagram: in this case, however, the path cannot be represented by means of a broken line, made up of piecemeal trades successively carried out during the same trade round, but must be represented by means of a set of independent trades, each connecting the same endowment allocation (the South-East corner of the Box, under the usual assumption of cornered traders) with allocations, denoted by x_t^S, with t running from t_0 to T_0, progressively approaching some allocation in the core, say $x_{T_0}^S$, where S stands for 'stationary' (see Fig. 3 above).

As shown above, as far as the "theory of simple contract" is concerned, no less than three alternative interpretations of the timing and nature of the trading process are put forward by Edgeworth over the years: the 'Marshallian', the 'mentalistic', and the 'stationary'. But only two of them, namely, the 'mentalistic' and the 'stationary', can survive in the context of the "theory of exchange" proper, to which we now turn.

As will be recalled, Edgeworth's basic conjecture is that both the extent of the competition among the traders and the determinatess of the outcome following from their interaction increase with the number of the traders participating in the economy. In order to prove his conjecture, Edgeworth contrives the device of evenly increasing the traders' number by repeatedly replicating the original Edgeworth Box economy. Namely, given an Edgeworth Box economy, $\mathcal{E}_E^{2\times 2}$, with two traders, respectively labelled A and B, each with specified characteristics (i.e., endowment and utility function), given any positive integer $n > 1$, an n-replica economy, ${}_n\mathcal{E}_E^{2\times 2}$, is an economy where there exist $2n$ traders, of which n, having the same characteristics as individual A, may be called type-A traders, while the remaining n, having the same characteristics as individual B, may be called type-B traders.

Within this framework, Edgeworth is able to (almost) prove three fundamental theorems which, in accordance with current usage, may be called the "Walrasian Equilibrium Is in the Core", the "Equal Treatment in the Core", and the "Shrinking Core" theorem (Varian, 1992, pp. 389-92); they respectively correspond to Debreu and Scarf's generalised versions of the original theorems, labelled as Theorems 1, 2, and 3 at pp. 240-3 of their 1963 paper.

As far as the "Walrasian Equilibrium Is in the Core" theorem, Edgeworth (1881, pp. 38-40, 112-6) can easily prove it with reference to the Edgeworth Box economy from which he starts. It is interesting to note that Edgeworth's proof heavily relies on the Walrasian apparatus of the "demand curves", appearing as "offer curves" in the Edgeworth's Box diagram (see Fig. 2 above): for, given that the Walrasian equilibrium (assumed unique) is identified by the intersection of the "offer curves" of the two traders, it necessarily satisfies both the conditions for Pareto optimality (equality of the marginal rates of substitution) and the individual rationality conditions, hence it belongs to the core of $\mathcal{E}_E^{2\times 2}$.

The extension of this result to an arbitrary n-replica economy, however, must be postponed until when the "Equal Treatment in the Core" theorem has been proven. This, in effect, is the first theorem to be discussed in the framework of replicated economies. In this regard, the formal proof put forward by Edgeworth (1881, p. 35) with reference to a 2-replica economy, ${}_2\mathcal{E}_E^{2\times 2}$, can be easily generalised to an arbitrary n-replica economy, ${}_n\mathcal{E}_E^{2\times 2}$. Though not providing the required generalisation, Edgeworth takes anyhow for granted the result. By so doing, he feels justified in extending the use of both the contract-curve equation and the geometrical apparatus referring to the original Edgeworth Box economy, $\mathcal{E}_E^{2\times 2}$, to any n-replica economy, ${}_n\mathcal{E}_E^{2\times 2}$, where n is an arbitrary positive integer greater than 1: to this end, in fact, it is enough to reinterpret the two traders of the original economy, A and B, as the representatives of their replicated sets or types, each type owning n identical traders, and similarly to reinterpret each two-trader allocation in the contract-curve of the original economy as the average two-type allocation in the contract-curve of the n-replica economy. By virtue of this reinterpretation, a suitably generalised Edgeworth Box diagram can be used at one and the same time to represent trades and allocations of individual traders as well as average trades and allocations of trader types. Further, since the conditions defining a competitive equilibrium in the two-trader economy are also satisfied in any replicated economy, the competitive equilibrium (assumed unique) of the original Edgeworth Box economy, $\mathcal{E}_E^{2\times 2}$, can be identified with the competitive equilibrium of any n-replica economy, ${}_n\mathcal{E}_E^{2\times 2}$, so that the "Walrasian Equilibrium Is in the Core" theorem is automatically generalised to all replicated economies. What holds for the competitive equilibrium construct, however, does not hold for the core construct.

Now the path is paved for proving the last and fundamental theorem, which, by confirming the shrinking of the core to the Walrasian equilibrium as the traders' number increases unboundedly, would also support Edgeworth's conjecture as to the relation between determinateness and size of the economy. Before proceeding to the proof of the "Shrinking Core" theorem, however, it must be pointed out that the results summarised in the previous paragraph crucially depend on how Edgeworth generalises the core concept, implicitly defined as the "set of final settlements" in the 2-trader framework of the Edgeworth Box economy, to the $2n$-trader framework of replicated economies, where n is an arbitrary positive integer greater than one.

Edgeworth's generalisation, as is well-known, rests on the "recontracting" mechanism, whose general rules are set at the beginning of *MP* (1881, pp. 16-20): in Edgeworth's view, it is precisely this mechanism which ensures the convergence of an economy with an arbitrary finite number of traders (greater than two) to some allocation in a suitably defined core. By resorting to the language of modern coalitional game theory, let us define a few concepts that are required for generalising the core concept to arbitrary n-replica economies. A coalition is a non-empty subset of traders. A coalition blocks an allocation if, by reallocating the aggregate coalition endowments among the coalition members, it obtains a coalition allocation, called blocking allocation, which ensures a utility level at least as great as that granted by the original allocation to all the coalition members, and a greater utility level to at least one of them. Finally, the core of an economy is the set of all the allocations that cannot be blocked by any coalition.

As we have seen, in the framework of an Edgeworth Box economy, Edgeworth suggests three alternative interpretations of the trading process, all of them satisfying the requirement that the process must converge to some allocation in the core of the two-trader economy (to some "final settlement", lying in the appropriate "portion of the contract-curve"). Yet, when it comes to discussing the issue of the convergence of the trading process to the core of an economy with a number of traders greater than two, Edgeworth saves just two of the original three interpretations, namely, the 'mentalistic' and the 'stationary', while dropping the third, that is, the 'Marshallian'. This is unmistakably shown by the following passage, meant to illustrate a bargaining process between "workmen and employers" over the employment conditions in the labour market:

> Two kinds of haggling may be distinguished as appropriate respectively to short and long periods. First, we may suppose the intending buyers and sellers to remain in communication without actually making exchanges [...]. By this preliminary tentative process a system of bargains complying with the condition of equilibrium is, as it were, rehearsed before it is actually performed. Or, second, one may suppose a performance to take place before such rehearsal is completed. On the first day in our example a set of hirings are made which prove not to be in accordance with the dispositions of the parties. These contracts terminating with the day, the parties encounter each other the following day[3] ([3] They recontract, in the phraseology of *Mathematical Psychics.*), with dispositions the same as on the first day, - like combatants *armis animisque refecti*, - in all respects as they were at the beginning of the first encounter, except that they have obtained by experience the knowledge that the system of bargains entered into on the first occasion does not fit the real dispositions of the parties. (Edgeworth, 1904, p. 40)

The first kind of "haggling" is nothing but a straightforward generalisation to an economy with many traders of the 'mentalistic' interpretation of the trading process already discussed above with reference to a two-trader economy. According to this interpretation, certainly the most popular among the interpreters of Edgeworth's thought[19], all contractual agreements entered in over the recontracting process are conditional, provisional, and revocable without penalty up until a stable solution (i.e., an allocation in the core of the economy concerned) is arrived at. Under such 'mentalistic' interpretation, therefore, recontracting becomes a purely virtual process in 'logical' time during which nothing observable can take place: for each trade round t, recontracting takes the nature of a "preliminary tentative process", closely resembling those "*tâtonnements* préliminaires" that, according to Walras (1988, p. 447, **4-5**), must be supposed to take place "en vue de l'établissement de l'équilibre en principe". No analytical or geometrical description should be provided of such a virtual process. Assuming the "Equal Treatment in the Core" theorem to hold, one might just imagine to plot in the generalised Edgeworth Box diagram (see again Fig. 3 above) the core allocation, say x_t^M, hopefully reached when the recontracting process attached to trade round t is eventually over.

The second kind of "haggling" is instead a laborious attempt to generalise to an economy with many traders the 'stationary' interpretation of the trading process already discussed above with reference to a two-trader economy. According to this interpretation, the recontracting process stretches over a sequence of trade rounds ("days" in Edgeworth's example), all characterised by the same data ("dispositions of the parties"). The contracts stipulated in each trade round are binding, irrevocable, and effective, but their material effects do not outlast the trade round itself; the only lasting consequences have to do with "knowlege" diffusion and learning "by experience". As before, one might

[19] See, e.g., Kaldor (1934 , p. 127) and Hicks (1934, p. 342, and 1939, p. 128).

think of illustrating such a process by plotting in a generalised Edgeworth Box diagram (see again Fig. 3 above) a sequence of allocations, say $\left(x_t^S\right)_{t=0,\ldots,T}$, each corresponding to the outcome of the contractual agreement stipulated in a specified trade round. But such illustration would be contrived, for at least two reasons: first, because no deterministic dynamical theory of the equilibration process is available, so that the illustration would be really unfounded; secondly, because no compelling interpretation can be offered for all the allocations x_t^S, with $t \neq T$, when the traders' number is greater than two: in that case, in fact, there is no reason to believe that all the traders of the same type behave in the same way out of the core, so that any allocation like x_t^S, with $t \neq T$, could at most be viewed as the average allocation resulting from unknown individual allocations.

It is worth noting that the few modern attempts at providing some openly dynamical version of the 'mentalistic' or 'stationary' interpretation of Edgeworth's recontracting process tend to preserve the characteristically undeterministic flavour of Edgeworth's original approach. Barring Uzawa's (1962), Negishi's (1962, pp. 660-3, and 1989, p. 294), and Madden's (1978) reinterpretations which, in spite of the chosen name ("Edgeworth's barter process"), are all but faithful to the original, one can easily verify that a version of the "Edgeworth process" much more true to the original, like the one put forward by Hahn (1982, pp. 772-3), while proving stability of the process, leaves both the path and the final allocation wholly unspecified. Finally, the reinterpretation of the recontracting process which is by far the closest to Edgeworth's original stance, namely, that suggested by Green (1975), is explicitly probabilistic in character.

While the 'mentalistic' and 'stationary' interpretations of the trading process in the Edgeworth Box economy can be generalised, with mixed success, to an economy with more than two traders, the 'Marshallian' interpretation cannot instead be so generalised. The reasons for this are simple. First, the 'Marshallian' interpretation is based on bilateral piecemeal trading. When there are more than two traders, it is not even clear what might mean to generalise the 'Marshallian' interpretation: if unlimited multilateral trading is allowed, as it happens with the standard interpretations of recontracting, the peculiarieties of the 'Marshallian' interpretation would seem to be lost; on the contrary, if only bilateral trading is allowed, then there is no guarantee that the trading process will converge to a Pareto-optimal allocation (Feldman 1973). Second, whatever the trading technology, if irrevocable trading is allowed for, then the competitive equilibrium necessarily becomes path-dependent, thereby impairing Edgeworth's "Shrinking Core" result.

Let us then finally turn to the "Shrinking Core" theorem. In order to prove it, Edgeworth (1881, pp. 35-7) proceeds as follows. First, by means of a geometrical argument based on a simple 2-replication of the original Edgeworth Box economy, he proves that “the points of the contract-curve in the immediate neighborhood of the limits [of that portion of the contract-curve that corresponds to the core of the original Edgeworth Box economy] cannot be final settlements", hence cannot belong to the core of the 2-replica economy, ${}_2\mathcal{E}_E^{2\times 2}$. Let us take, e.g., the rightmost allocation in the core of the Edgeworth Box economy, $\mathcal{E}_E^{2\times 2}$, which is here identified, for convenience, with the 1-replica economy, ${}_1\mathcal{E}_E^{2\times 2}$; in Fig. 3 this allocation is denoted by $x^{1r} = \left(x_A^{1r}, x_B^{1r}\right)$, where the superscripts 1 and r stand for '1-replica economy' and 'rightmost', respectively. Let us now consider the 2-replica economy, ${}_2\mathcal{E}_E^{2\times 2}$, which owns in the whole 4 traders: precisely, 2 identical type-A traders, denoted by A_1 and A_2, and 2 identical type-B traders, denoted by B_1 and B_2. The x^{1r} allocation can now be reinterpreted as a type-representative, equal-treatment allocation in the 2-replica economy, that is, $x^{1r} = \left(x_{A_1}^{1r} = x_{A_2}^{1r}, x_{B_1}^{1r} = x_{B_2}^{1r}\right)$. Yet, by exploiting the strict quasi-concavity of the utility functions, Edgeworth is able to show that a coalition formed by the two type-B traders, B_1 and B_2, who are relatively "disadvantaged" in the x^{1r} allocation, and one of the relatively "advantaged" type-A traders, say A_1, can block the x^{1r} allocation by reallocating the coalition endowments in such a way as to get the blocking allocation $\left(x_{A_1}^{2b}, x_{B_1}^{2b}, x_{B_2}^{2b}\right)$, satisfying $x_{B_1}^{2b} = x_{B_2}^{2b}$ and $x_{A_1}^{2b} + x_{B_1}^{2b} + x_{B_2}^{2b} = \omega_{A_1}^{2b} + \omega_{B_1}^{2b} + \omega_{B_2}^{2b}$, where the superscripts 2 and b stand for '2-replica economy' and 'beginning of the blocking process', respectively; the other type-A trader, A_2, would be simply left with his endowment, that is, $x_{A_2}^{2b} = \omega_{A_2}$.

Yet, the resulting 4-trader allocation in the 2-replica economy, $\left(x_{A_1}^{2b}, x_{A_2}^{2b}, x_{B_1}^{2b}, x_{B_2}^{2b}\right)$, which does not lie on the contract-curve and does not satisfy the equal treatment property (as far as type-A traders are concerned), would be blocked by some coalition; the recontracting process would go on until a new allocation located in the portion of the contract-curve to the left of x^{1r} would be reached. However, all the allocations lying on the portion of the contract-curve between x^{1r} and x^{2r} would be eliminated by the same blocking procedure as that by means of which x^{1r} was initially eliminated; the reason why $x^{2r} = \left(x_{A_1}^{2r}, x_{A_2}^{2r}, x_{B_1}^{2r}, x_{B_2}^{2r}\right)$ can no longer be eliminated in this way should be clear from the diagram in Fig. 3, where the superscript e in the allocation $\left(x_{A_1}^{2e}, x_{A_2}^{2e}, x_{B_1}^{2e}, x_{B_2}^{2e}\right)$ stands for 'end of the blocking process'. A similar process would have taken place if one had started from the leftmost allocation x^{1l}: in a 2-replica economy, all the allocations lying in the portion of the contract-curve between x^{1l} and x^{2l} would be eliminated by the same blocking procedure as before. The set of the allocations in the contract-curve which cannot be eliminated in this way, that is, the set of the allocations lying in the portion of the contract-curve between x^{2l} and x^{2r}, extremes included, is the core of the 2-replica economy, or the 2-core, which is strictly contained in the original core, or the 1-core.

Consider now an allocation in the contract curve such that the straight line segment connecting it with the endowment allocation (the South-East corner of the Edgeworth Box diagram) cuts the indifference curves passing through it. Any such allocation can be eliminated by a sufficiently large blocking coalition, by means of a suitably generalised version of the procedure suggested by Edgeworth with reference to a 2-replica economy. As the replication process proceeds, larger and larger blocking coalitions can be formed, so that more and more allocations in the contract-curve can be eliminated by means of Edgeworth's blocking procedure. Hence, the replication process generates a nested sequence of cores converging towards the Walrasian (or Jevonsian) equilibrium allocation x^W (or, what is the same, x^J): in fact, since "the common tangent to both indifference-curves at the point x^W [or x^J] is the vector from the [the endowment allocation]", the Walrasian (or Jevonsian) equilibrium allocation is the only allocation in the contract-curve which survives Edgeworth's elimination process when the number of replications, hence of traders, grows unboundedly large. This is enough to prove the "Shrinking core" theorem (Edgeworth 1881, p. 38).

Edgeworth's last remark also suggests which are, in his opinion, the truly distinguishing features of the Walrasian (or Jevonsian) equilibrium allocation: among all the "final settlements" in the original core, the equilibrium allocation is the only one that can be reached by means of one single trade, carried out at a uniform rate of exchange, coinciding with the Walrasian equilibrium "price" or the Jevonsian equilibrium "ratio of exchange"; such properties, however, hold only when the number of traders is "practically infinite", so that "perfect competition" rules. This conclusion allows one to assess Edgeworth's stance with respect to Jevons's Law of Indifference and Walras's Law of One Price.

As far as Jevons's Law of Indifference is concerned, Edgeworth rediscovers and endorses it, turning it into a central ingredient of his theory, with one important qualification. As in Jevons, so in Edgeworth, when the Law of Indifference rules, the path from the endowment allocation to the equilibrium allocation can be travelled at one stroke, at a uniform rate of exchange, and even "instantaneously", at least if the 'mentalistic' interpretation of recontracting is embraced. Likewise, price uniformity holds at equilibrium only, while the equilibration process is left, or even must be left, in the dark. Unlike Jevons, however, Edgeworth strictly confines the validity of the Law of Indifference to large economies only, explicitly introducing a mechanism which, by combining replication and recontracting, explains why a "practically infinite" number of traders is required for the Law to hold true.

The above remarks also help understand why Edgeworth rejects Walras's extensive interpretation and wide-ranging use of the Law of One Price. First, as we have seen, for Edgeworth price uniformity is a perfectly competitive equilibrium phenomenon only, so that assuming it in a disequilibrium context, as Walras does in his *tâtonnement* contruct, is wholly unjustified (Edgeworth 1881, pp. 30-1, 47-8, 109, 116 fn. 1, 143 fn. 1; 1891a, p. 367, fn. 1; 1910, p. 374; 1915, p. 453; 1925b, p. 312). Secondly, price uniformity ought not to be "postulated", as Walras does (Edgeworth 1881, p. 40), but made to descend from some more basic principle, such as recontracting:

> Walras's laboured description of prices set up or "cried" in the market is calculated to divert attention from a sort of higgling which may be regarded as more fundamental than his conception, the process of recontract as described in these pages and in an earlier essay [Edgeworth 1925d, partial English "translation" of 1891b]. It is believed to be a more elementary manifestation of the propensity to truck than

> even the effort to buy in the cheapest and sell in the dearest market. The proposition that there is only one price in a perfect market may be regarded as deducible from the more axiomatic principle of recontract (*Mathematical Psychics*, p. 40 and context). (Edgeworth 1925b, pp. 311-2).

Edgeworth is aware that in Walrasian general equilibrium theory the existence of a "price system" is not regarded "as axiomatic, rather as deducible (by way of "arbitrage") from a greater number [of] rates-of-exchange" . Yet, he immediately qualifies his acknowledgment by adding that

> even the existence of a uniform rate-of-exchange between any two commodities is perhaps not so much axiomatic as deducible from the process of competition in a perfect market. (Edgeworth, 1915, p. 453)

This means that, for Edgeworth, Walras's use of Cournot's theory of arbitrage in deriving his "perfect" or "general equilibrium" conditions from the "imperfect equilibrium" conditions provisionally holding on the "special markets" for pairwise direct exchanges of commodities, though a step in the right direction, is still not enough: for a complete explanation would require to avoid postulating uniform prices in the "special markets", too,trying instead to deduce them from some more fundamental primitives. But, as explained at the end of the previous Section, this is a goal that Walras does not even start to pursue.

5 Negishi on Jevons, Walras, and Edgeworth

In his 1982 paper, Negishi aims at vindicating Jevons's Law of Indifference by showing that, though not so powerful as implicitly suggested by Jevons himself, the Law is more powerful than conceded by Walras and Edgeworth, provided that its underlying arbitrage mechanism is duly taken into account. Negishi's stance on Jevons's and Walras's contributions on this issue is similar to that taken by Edgeworth, with a difference: while for Edgeworth Jevons's Law of Indifference holds only in large economies, so that a "practically infinite" number of traders is required for price uniformity and equilibrium establishment, for Negishi no large numbers are necessary for the Law to hold, a uniform price to rule, and a Walrasian or Jevonsian equilibrium to obtain. Unlike Jevons, for whom two traders are apparently enough for equilibrium establishment, an Edgeworth Box economy is not sufficient for Negishi, for in such an economy there would be no room for arbitrage; but a slightly larger economy, such as an Edgeworthian 2-replica economy, with only four, pairwise identical traders, would do, for with more than two traders arbitrage becomes possible.

In developing his argument, Negishi largely relies on Edgeworth's conceptual system, even if he freely builds upon the foundations laid by the latter. Let us then consider a 2-replica economy, plotting its curves and variables in a standard

Edgeworth Box diagram (Fig. 4 below). Let x^{1l}, x^{1r}, x^{2l}, x^{2r}, x^J, and x^W be interpreted as before. Further, let x_2^2 be an allocation lying in the portion of the contract curve between x^{2l} and x^J, with $x_2^2 \neq x^J$. According to Edgeworth, such allocation would belong to the 2-core, for it could not be blocked by means of the standard recontracting mechanism at work in a 2-replica economy. Since a number of traders greater than 4 would be required to block an allocation like x_2^2, in a 2-replica economy it would tend to persist, once arrived at by whatever route, for want of an effective elimination mechanism.

Figure 4 about here

Edgeworth's result is quite standard. Yet Negishi disputes it, for, according to him, it follows from Edgeworth's excessively conservative interpretation of the coalition concept and the associated recontracting construct. For Edgeworth, as we have seen, a coalition is simply a non-empty subset of traders; a coalition is said to block an allocation when, by reallocating the resources of its own members only, can produce an outcome that is weakly preferable for all its members, and them only, to that implied by the original allocation. For Negishi (1982, p. 228), howevere, Edgeworth's coalitions are just "closed" or "pure" coalitions, for they take into accont resources and welfare of their members only; but since the real world is full of instances of "open" or "impure" coalitions, where the boundaries between coalition members and non-members are blurred, the theory should draw its inspiration from reality and learn to employ a wider concept of coalition, too.

Keeping this in mind, let us now consider how the allocation x_2^2 might have been arrived at. Let us approach the problem step by step. To start with, let us suppose that the economy under discussion is not a 2-replica economy, but an Edgeworth Box economy, as in Edgeworth's "theory of simple contract"; furthermore, let us interpret the trading process in the 'Marshallian' way, that is, let us view it as a sequence of piecemeal bilateral trades taking place over the same trade round. In such a case, we would be forced to conclude that the allocation x_2^2, lying in the contract-curve, but different from the Walrasian equilibrium, must be reached by a path involving at least two partial piecemeal trades associated with different rates of exchange. Even if there exists an infinite number of paths potentially leading the two-trader economy to x_2^2, let us suppose, for simplicity, that the actual path consists just of two piecemeal successive trades, the first leading the traders from ω to x_1^2 and the second from x_1^2 to x_2^2, taking place at the respective rates of exchange r_{21}^1 and r_{21}^2, with $r_{21}^1 < r_{21}^2$ (see Fig. 4 above).

Up to this point, we would just be following in Edgeworth's steps, borrowing especially from the first part of his paper in Italian on Marshall's theory of barter (1891b). Yet, at this point, Negishi parts company with Edgeworth, for he suggests to interpret the same broken line from ω to x_2^2, via x_1^2, as the path travelled by the four, pairwise identical traders (A_1 and A_2, B_1 and B_2) belonging to the 2-replica economy from which we started: Negishi's idea is that A_1 trades with B_1, A_2 with B_2, and that the respective successive piecemeal trades of the two trading pairs are the same, so that ω, x_1^2 and x_2^2 should now be interpreted as the two-component allocations of the representatives of the two trader-types, that is, $\omega = (\omega_{A_i}, \omega_{B_i})$, $x_1^2 = \left(x_{1A_i}^2, x_{1B_i}^2\right)$ and $x_2^2 = \left(x_{2A_i}^2, x_{2B_i}^2\right)$, for $i = 1, 2$. However, Negishi's proposed switch from Edgeworth's "theory of simple contract" (two traders only) to his "theory of exchange" proper (more than two traders) is unwarranted. As we have seen in the previous Section, in fact, when there are more than two traders, the 'Marshallian' interpretation of the trading process as a sequence of observable, irrevocable, piecemeal, bilateral trade moves within the same trade round must be abandoned, according to Edgeworth, in favour of one of the other two possible interpretations of the trading process: the 'mentalistic', where the unobservable bargainings are supposed to take place before any actual trade occurs in a given trade round, and the 'stationary', where the irrevocable contracts are supposed to be stipulated over a sequence of trade rounds. But, in the first case, no specification of the trading process should be attempted, according to Edgeworth; in the second, instead, one might try, with some effort and dubious results, to represent a sequence of observable average allocations, resulting however from contracts signed in different trade rounds, rather than the same.

Hence, with his suggested interpretation of the broken line in Fig. 4, Negishi is trying to mix together some features of Edgeworth's "theory of simple contract" with other aspects pertaining to his "theory of exchange" proper. Yet, the resulting mixture is really a muddle, as can be seen from Negishi's hesitations and ambiguities about the nature of the contracts appearing in his story: on some occasions they are supposed to be "provisional" and revocable, hence presumably simultaneous (or anyhow preliminary) and unobservable; on other occasions, however, they are supposed to be "successive" and capable of being specified in detail, hence presumably observable (Negishi 1982, pp. 225-6).

Putting provisionally aside these difficulties, let us now see how allocation x_2^2, unblockable in a 2-replica economy according to Edgeworth, can instead be blocked according to Negishi. The basic idea is that what cannot be done by Edgeworth's "closed" coalitions, can instead be accomplished by Negishi's "open" coalitions. For example, let us suppose that A_2 and B_1 form an "open" coalition $\{A_2^o, B_1^o\}$, meaning by this that, beyond fully relying on their own resources, A_2^o and B_1^o also keep some "links" with the other two traders, A_1 and B_2, into whose resources they also have some limited chance to tap. In order to block the allocation x_2^2, A_2^o and B_1^o proceed as follows: B_1^o cancels part of his 'second' contract with A_1 at the unfavourable (for him) rate of exchange r_{21}^2, while keeping the rest of this contract and the whole of his 'first' contract with A_1 at the favourable (for him) rate of exchange r_{21}^1; A_2^o cancels part of her 'first' contract with B_2 at the unfavourable (for her) rate of exchange r_{21}^1, while accepting to offset the suppressed part of her contract with B_2 by stipulating a new contract with B_1^o at identical terms, but in the opposite direction (this last assumption being made only for simplicity, for any rate of exchange in the open interval $\left(r_{21}^1, r_{21}^2\right)$ would make both members of the "open" coalition better off). The outcome of this sort of revised recontracting proposal is shown in Fig. 4, where $x_{A_1}^N$, $x_{A_2}^N$, $x_{B_1}^N$, and $x_{B_2}^N$ are the final consumption bundles of the four traders, measured with respect to the appropriate individual coordinate axes, at the end of the process imagined by Negishi in his example (the superscript N standing for Negishi). The final allocation of the "open" coalition, $\left(x_{A_2}^N, x_{B_1}^N\right)$, is weakly advantageous for its members with respect to the original one, $\left(x_{2A_2}^2, x_{2B_1}^2\right)$, for $u_{A_2}\left(x_{A_2}^N\right) = u_{A_2}\left(x_{2A_2}^2\right)$ and $u_{B_1}\left(x_{B_1}^N\right) > u_{B_1}\left(x_{2B_1}^2\right)$. Hence, in Negishi's opinion, the open coalition $\{A_2^o, B_1^o\}$ can block the 2-core allocation x_2^2.

This sort of revised elimination procedure, based on Negishi's wider coalition concept and enlarged recontracting construct, applies to all allocations in the contract-curve different from the Walrasian or Jevonsian equilibrium allocation, x^W or x^J. The reason for this, according to Negishi, is simple: the elimination procedure rests on the exploitation of arbitrage opportunities, whose very existence depends on the co-existence of different rates of exchange at one and the same time; but while all the allocations in the contract-curve different from the equilibrium allocation must be reached in at least two steps, each associated with a specific rate of exchange, so that arbitrage opportunities do exist in this case, the equilibrium allocation, instead, is arrived at in one single step, associated with one single rate of exchange, so that no arbitrage opportunity can arise here. Hence, while all the allocations other than the equilibrium one would succumb to the revised elimination procedure, the Walrasian or Jevonsian equilibrium allocation would live through it. Contrary to Edgeworth's conclusions, therefore, price uniformity and equilibrium establishment would not require a "practically infinite" number of traders: Jevons's Law of Indifference would assert itself in a finite economy, too, provided that the traders' number were greater than two, so that arbitrage could display its strength.

The above story, however, is seriously faulty. In the first place, the issue of the timing of the trading process, provisionally set aside above, must now be taken up again. In Negishi's example four piecemeal bilateral trades are supposed to take place to start with, involving two pairs of traders (either A_1 and B_1, or A_2 and B_2) and two steps (say, step 1 and step 2): for each pair of traders there are two trades at different rates of exchange, one for each step; likewise, for each step there are two trades at the same rate of exchange, one for each pair of traders. Now, at each step there can be no arbitrage, since the rate of exchange is the same for the two trades occurring at the same step. So, one is led to think of arbitraging activities between different steps, since the rates of exchange differ as between trades occurring at step 1 and 2, respectively. Yet, if the steps are thought of as "successive" and the contracts as irrevocable, how can arbitrage occur? On the contrary, if the steps are thought of as simultaneous and the contracts as "provisional", how is it possible that the same pair of traders should decide to stipulate at one and the same time two "provisional" contracts for trading the same two commodities at different rates of exchange? It is clear that Negishi's assumptions about the trading technology and the traders' arbitraging activities should be significantly perfected in order to make the story palatable.

Yet a second objection, concerning the very concept of an "open" coalition, is even more fundamental. Going back to Negishi's example, one can easily check that the true members of the "open" coalition, A_2^o and B_1^o, can (weakly) increase their utility levels only by "exploiting" the other two traders, A_1 and B_2, with whom they keep some "links": in effect, the utility levels of the other two traders necessarily decrease as a result of the revised recontracting proposal of the "open" coalition members, namely, $u_{A_1}\left(x_{A_1}^N\right) < u_{A_1}\left(x_{2A_1}^2\right)$ and $u_{B_2}\left(x_{B_2}^N\right) < u_{B_2}\left(x_{2B_2}^2\right)$. Yet, since an "open" coalition can improve the welfare of its members only at the expenses of the welfare of non-members, Negishi's concept of an "open" coalition immediately appears as much weaker than Edgeworth's "closed" coalition concept, for in a "closed" coalition all improvement can only come from the reallocation of the resources owned by the coalition members. Negishi (1882, p. 228) recognises that an "open" coalition is less stable than a "closed" one, but he adds that "the stability of the realised coalition itself is not required to block an allocation in Edgeworth's exchange game". Yet, if Negishi is right in pointing out that the stability of any realised coalition is not necessary for blocking, he is certainly wrong when he forgets that the credibility of the threat made by a blocking coalition, which rests precisely on the autarchic character of Edgeworth's "closed" coalitions, is essential for blocking. Moreover, "open" coalitions are supposed to rely on the unawareness of the cheated partners (Negishi 1882, p. 228). Yet, if cheating is allowed for, and assumed not to be spotted, provided that the extent of the swindle is limited, then even a Walrasian or Jevonsian equilibrium allocation would not be immune from the destabilising power of "open" coalitions. In the end, therefore, Negishi's approach appears to oscillate between proving too little (for the threats of "open" coalitions are not credible, hence no allocation can be blocked in this way) and proving too much (for, if cheating is allowed for and deemed effective, then any allocation can be destabilised).

To sum up, Negishi's attempt to use Edgeworth's machinery (freely revised) to vindicate the effectiveness of Jevons's Law of Indifference in small economies is unsuccessful. Moreover, his attempt shows that Edgeworth's assumptions (in particular, those concerning the nature and timing of the trading process and the impossibility of a general deterministic dynamical theory of the equilibration process) are carefully chosen and cannot be easily replaced without incurring into serious mistakes or unfounded claims.

6 Concluding remarks

Jevons, Walras, and Edgeworth develop their pure-exchange equilibrium models over the decade 1871-1881. All three of them make use of some version of a Law, called Law of Indifference by Jevons and Edgeworth and often referred to as the Law of One Price in connection with Walrasian economics. About one century later, Negishi resumes the time-honoured discussion about the Law, taking an unconventional stance. Since the relations among the equilibrium concepts, the equilibration mechanisms, and the interpretations of the Law put forward by Jevons, Walras, and Edgeworth form a highly complicated pattern, the first aim pursued in this paper has been to review the three economists' quite different approaches, identifying their peculiar uses of the Law in their respective equilibrium and equilibration theories (if any). Secondly, we have also tried to assess the robustness of Negishi's more recent attempt at proving that, contary to Edgeworth's original conjecture, a competitive equilibrium can be attained even in small economies, provided that the true driving force underlying Jevons's Law of Indifference, namely, its implicit arbitrage mechanism, is allowed to operate and to carry its effects through.

Starting from Jevons, we have shown that his attempts at empirically and theoretically justifying the Law of Indifference end up turning it into a quasi-tautology. This, however, has the undesirable side-effect of killing any analysis of the equilibration process, forcing an instantaneous interpretation of Jevons's equilibrium concept, and confining the consistent application of his theory to a two-trader, two-commodity economy only.

Such limitations do not constrain Walras's equilibrium concept, which is much more general than Jevons's. Yet, also Walras, though possessing a partially developed theory of arbitrage based on Cournot's approach, is unable to found his Law of One Price upon such a theory, or any other theory of arbitrage, for that matter. Therefore, Walras is forced to postulate the Law of One Price as an axiom. As an axiom, however, the Law plays a paramount role in Walras's theoretical system, underlying both his equilibrium theory and his analysis of the equilibration processes. As regards equilibration, in particular, the Law provides the building block on which the Walrasian *tâtonnement* construct is raised.

Edgeworth, writing after Jevons and Walras, can and does explicitly refer to their theories, either to praise or to criticise them. He starts from Jevons, vindicating his interpretation of the Law of Indifference, viewed as an equilibrium property. Unlike Jevons, however, Edgeworth denies the validity of the Law in all economies where the traders' number is short of the "practically infinite". He also avoids turning the Law into a tautology or postulating it as an axiom. On the contrary, he tells a plausible story, based on the joint operation of his replication and recontracting mechanisms, a story meant to support the emergence of the Law, as well as the convergence of the economy towards a Walrasian or Jevonsian equilibrium, as the traders' number grows unboundedly. Finally, he rejects Walras's use of the Law of One Price in his analysis of equilibration processes, asserting that no price uniformity can hold out of perfectly competitive equilibrium and explicitly maintaining that the Edgeworthian principle of recontracting is "more fundamental" than the Walrasian principle of *tâtonnement*.

Lastly, we have critically examined Negishi's contribution. Negishi tries to strengthen Edgeworth's results: precisely, by exploiting the mechanism of arbitrage, regarded as the driving force behind Jevons's Law of Indifference or Walras's Law of One Price, he tries to prove that such Law, viewed as an equilibrium property, holds also in finite economies, so that convergence to a Jevonsian or Walrasian equilibrium is instantaneously assured, via arbitrage, even in economies with a small number of traders. Yet, Negishi's proof is unconvincing, for he tries to exploit Edgeworth's machinery without preserving the latter's carefully chosen assumptions and qualifications. Moreover, Negishi's concept of an "open" coalition, meant to generalise Edgeworth's concept of a "closed" coalition, is so weak and inconsistent that its associated blocking mechanism is wholly untrustworthy.

The failure of Negishi's attempt confirms, in the last analysis, that Edgeworth's results cannot be improved upon, so long as one accepts, as Negishi does, to keep the analysis at the level of abstraction and generality chosen by Edgeworth. Stronger results may be hoped for only by further specifying the trading technology, the bargaining machinery, the information transmission mechanism, or any other property of the trading or contracting process; but this, of course, would make the hopefully stronger results depend on the special assumptions adopted.

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