Is the Leontief input-output model a production-prices model?

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Abstract

The Leontief model is generally considered as belonging to the category of the "production-prices models", aside of the Ricardian, Marxian and Sraffaian models. This paper clearly demonstrates that this is a superficial point of view. As the Leontief model handles price indexes, it is over two periods: the Leontief price indexes solve the model for the base time period. The production-prices model that corresponds to it is over one period: the current prices solve the model for the current time period. Both models diverge generally unless a very strong assumption is done: the interindustry matrix of direct and indirect quantities of labor incorporated per unit of physical output—the interindustry matrix of Marxian values—is stable over time; this implies that the vertically integrated coefficients of labor are stable or that the physical technical coefficients and the physical coefficients of labor are stable over time, two very strong assumptions. We conclude that the Leontief model is generally not a production-prices model, unless quite Classical-Marxian assumptions are done.

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1 Introduction

The Leontief model writes as

$$\begin{cases} \mathbf{A}\mathbf{x} + \mathbf{f} = \mathbf{x} \\ \boldsymbol{\rho}'\mathbf{A} + \boldsymbol{\rho}_v'\mathbf{L} = \boldsymbol{\rho}' \end{cases}$$

where $\mathbf{x}$ is the vector of total outputs, $\mathbf{f}$ the vector of final demands, $\mathbf{A}$ the matrix of technical coefficients, $\mathbf{L} = \hat{\mathbf{v}}\,[\hat{\mathbf{x}}]^{-1}$ the diagonal matrix of labor coefficients (where $\mathbf{v}$ is the vector of quantities of labor), $\boldsymbol{\rho}$ the vector of "prices" and $\boldsymbol{\rho}_v$ the vector of labor "prices", the prime denoting the transposed of vector; e.g. $\mathbf{s}'$ is the transposed of $\mathbf{s}$); the terms "prices" are in inverted commas because their status has to be discussed. This model solves as

$$\begin{cases} \mathbf{x} = [\mathbf{I} - \mathbf{A}]^{-1}\,\mathbf{f} \\ \boldsymbol{\rho}' = \boldsymbol{\rho}_v'\mathbf{L}\,[\mathbf{I} - \mathbf{A}]^{-1} \end{cases}$$

One of the main hypotheses of the Leontief model is that it uses monetary data (i.e., in currency units) instead of physical data, a stroke of genius! This hypothesis allows to aggregate different commodities (e.g. "Steel" and "Energy") along a column of the input-output table in addition to the obvious aggregation per rows; it allows to reduce the number of sectors and products by aggregation; the production coefficients can be aggregated per columns (and their sum is lower than one). This makes the input-output model very handy to use in applied studies, explains its success and justifies the "Nobel Prize": the real data extracted from national accounting are compiled in monetary terms.

Even if the Leontief input-output model was considered by his author as a miniature General Equilibrium, that is, a neoclassical model,[1] it is generally considered as belonging to the category of the "production-prices models", a category where Ricardo, Marx and Sraffa are the main authors and which is qualified as Classical (see Pasinetti 1977). However, the production-prices model that corresponds to it is

$$\begin{cases} \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{f}} = \bar{\mathbf{x}} \\ \mathbf{p}'\bar{\mathbf{A}} + \mathbf{p}_v'\bar{\mathbf{L}} = \mathbf{p}' \end{cases}$$

where $\mathbf{p}$ denotes the output price vector, $\mathbf{p}_v$ the labor price vector and the other notations are similar except that the bar denotes physical units. This model

[1] Even if the Leontief production function—which uses complementary inputs—can be considered as a particular case of the Neoclassical production functions, the Leontief input-output model does not use the utility functions that clearly watermarked the Neoclassical model.

solves as

$$\begin{cases} \bar{\mathbf{x}} = \left[\mathbf{I} - \bar{\mathbf{A}}\right]^{-1} \bar{\mathbf{f}} \\ \mathbf{p}' = \mathbf{p}'_v \bar{\mathbf{L}} \left[\mathbf{I} - \bar{\mathbf{A}}\right]^{-1} \end{cases}$$

While the Leontief model is able to open out onto real-world applications, the purpose of a production-prices model is largely theoretical, which is completely different: a production-prices model is a theoretical reference unable to generate applications easily because of the heterogeneity between commodities which makes that the input-output tables cannot be compiled from physical data. This is why one could wonder under which conditions they can be equivalent, except the passage from currency units to physical units and vice versa.

Moreover, the production-prices model finds prices as solution but the solution of the Leontief model cannot be prices: it is actually price indexes, which is different. Replacing prices by price indexes is not innocent: it implies that exists a "base time period"[2] that serves as common reference for all price indexes for deflating monetary tables in order to remove price effects. We will show that considering price indexes makes that the Leontief model uses tow periods, with technical coefficients determined at the current period on one hand, and price indexes determined by the base period (compared to the current period) on the other hand. In other words, there is a time shifting in the Leontief model, while in a production-prices model, production coefficients and prices pertain to the same time period. Both types of models seem completely incompatible. This is why in this paper we will examine under which conditions the Leontief model can be considered as a production-prices model.

The paper is organized as follows. Section 2 formally derives the Leontief model and the corresponding production-prices model. Section 3 examines under which conditions the both models coincide. Section 4 concludes and section 1 is this introduction.

2 Formal derivation of the models

2.1 The accounting balances

2.1.1 Physical closure

The model should be physically closed: the total of sales is equal to the total output, in physical quantities. Therefore, for each commodity i and each seller

[2] The time period is often the year in national accounting.

i, the total of what is sold to the buyers (including i) is equal to the total of what can be sold.[3] In matrix terms, this is:

$$\bar{\mathbf{Z}}\mathbf{s} + \bar{\mathbf{f}} = \bar{\mathbf{x}} \tag{1}$$

Closeness is not a strong assumption: it is sufficient to consider that each buyer i buys itself all that has not be bought by the others. One deduces of (1) that $\bar{\mathbf{x}} > 0$.

2.1.2 Money closure

Even if the model is physical, commodities and labor in each sector j have prices, respectively denoted $\mathbf{p}$ and $\mathbf{p}_v$. The model is doubly closed. First, for each seller i, the total value of what is sold to all buyers (including i) is equal to the total value of what can be sold, which is the the same than the physical closure. In matrix terms, this is $\hat{\mathbf{p}}\bar{\mathbf{Z}}\mathbf{s} + \hat{\mathbf{p}}\bar{\mathbf{f}} = \hat{\mathbf{p}}\bar{\mathbf{x}} \Leftrightarrow (1)$

$$\Leftrightarrow \mathbf{Z}\mathbf{s} + \mathbf{f} = \mathbf{x} \tag{2}$$

where $\mathbf{Z} \equiv \hat{\mathbf{p}}\bar{\mathbf{Z}}$ and $\mathbf{f} \equiv \hat{\mathbf{p}}\bar{\mathbf{f}}$, the hat (e.g. in $\hat{\mathbf{p}}$) denoting the diagonal matrix formed from a vector (e.g. $\mathbf{p}$).

Second, for each buyer j, the total value of what is bought to all sellers (including j) is equal to the total value of what can be bought. In matrix terms, this is $\mathbf{p}'\bar{\mathbf{Z}} + \mathbf{p}'_{\mathbf{v}}\hat{\bar{\mathbf{v}}} = \mathbf{p}'\bar{\mathbf{x}}$

$$\Leftrightarrow \mathbf{s}'\mathbf{Z} + \mathbf{v}' = \mathbf{x}' \tag{3}$$

where $\bar{\mathbf{v}}$ is the vector of the physical quantities of labor (or of any other factor) used by each sector and $\mathbf{v} = \hat{\mathbf{p}}_v\bar{\mathbf{v}}$.[4] Any model, Leontief model or production prices model of similar structure, should follow the balances (2) and (3) (the physical balance (1) is redundant with (2)).

[3] We limit all the discussions to the familiar "open" model where the final demand is exogenous and there is only one exogenous factor; however, they can be transposed to the "closed" model easily.

[4] As the price of labor varies across sectors, there is implicitly either an assumption that labor is heterogeneous (the labor employed by one sector not being the same as the labor employed by another sector), or an assumption that labor is immobile (otherwise the price of labor would be uniform among sectors because it would transfer from low-paying to high-paying sectors), or an assumption that working conditions across different sectors are heterogeneous. Anyway, it is more general (but a little more complicated) to consider that the price of labor varies across sectors: a uniform price can always be retrieved by setting $p_{v_j} = p_v$ for all j.

2.2 The production-prices model à la Leontief

It is necessary to recall how is derived the production-prices model à la Leontief (with final demand and one factor of production and a single technique with complementary inputs, not a general production prices model à la Sraffa). In this model, quantities and prices are explicitly considered.

2.2.1 The production coefficients

It is assumed that each sector buys each commodity in fixed proportions following the *production coefficients* defined in physical terms and assumed stable:[5]

$$\bar{\mathbf{A}} = \bar{\mathbf{Z}} \left[\hat{\bar{\mathbf{x}}}\right]^{-1} \tag{4}$$

2.2.2 The model's equations

In the *primal*, when (4) is included, the balance (2) turns out to be $\sum_{j=1}^{n} \bar{a}_{ij}\bar{x}_j + \bar{\bar{f}}_i = \bar{x}_i$ for any i, that is:

$$\bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{f}} = \bar{\mathbf{x}} \tag{5}$$

One remarks that the solution $\bar{\mathbf{x}} = \left[\mathbf{I} - \bar{\mathbf{A}}\right]^{-1} \bar{\mathbf{f}}$ is expressed in physical terms. In the *dual*, when (4) is included, the balance (3) turns out to be

$$\mathbf{p}'\bar{\mathbf{A}} + \mathbf{p}'_v\bar{\mathbf{L}} = \mathbf{p}' \tag{6}$$

where the matrix $\bar{\mathbf{L}} = \hat{\mathbf{v}} \left[\hat{\bar{\mathbf{x}}}\right]^{-1}$ is the diagonal matrix of labor coefficients $\bar{l}_i = \frac{\bar{v}_i}{\bar{x}_i}$. Therefore, the prices come naturally as a function of the input labor coefficients multiplied by factor's price:

$$\mathbf{p}' = \mathbf{p}'_v\bar{\mathbf{L}} \left(\mathbf{I} - \bar{\mathbf{A}}\right)^{-1} \tag{7}$$

2.3 The Leontief model

In the Leontief model, physical quantities are never introduced. We assume that the set of commodities is the same in both production-prices and input-output models.[6] The balances (2) and (3) still hold. *Technical coefficients* are defined

[5] Remember that nothing prevents the coefficients from being higher than one: their magnitude depends of what scale is chosen but this has no impact on the solution as the determinant $\left|\mathbf{I} - \bar{\mathbf{A}}\right|$ is scale-independent, as the result, after appropriate conversion of scale.

[6] Actually, this is not granted in practice. On the one hand, there could be more commodities in monetary input-output tables because some immaterial commodities are impossible to capture in physical terms. On the other hand, there could be more commodities in physical

in monetary terms in currency units and assumed stable (Leontief 1970, 1985; Miller and Blair 2009):

$$\mathbf{A} = \mathbf{Z}\left[\hat{\mathbf{x}}\right]^{-1} \tag{8}$$

2.3.1 The primal

Solving the primal—by rows—generates no difficulties. Combining (8) into equation (2) implies $\sum_{j=1}^{n} a_{ij}x_i + f_i = x_i$ for any i, that is,

$$\mathbf{Ax} + \mathbf{f} = \mathbf{x} \tag{9}$$

The outputs in currency units are now deduced from final demands in currency units, that is,

$$\mathbf{x} = \left[\mathbf{I} - \mathbf{A}\right]^{-1}\mathbf{f}$$

This primal poses no difficulties. It is not the case of the dual.

2.3.2 The dual

Introducing the index prices Denote by $\mathbf{L} = \hat{\mathbf{v}}\left[\hat{\mathbf{x}}\right]^{-1}$ the matrix of coefficients $l_i = \frac{v_i}{x_i}$. If we directly introduce the technical coefficients (8) into the balance (3), this one turns out to be an identity: $\sum_{i=1}^{n} a_{ij} + \sum_{i=1}^{n} l_i = 1$ for any j, that is,

$$\mathbf{s}'\mathbf{A} + \mathbf{s}'\mathbf{L} \equiv \mathbf{s}' \tag{10}$$

Hence, the dual Leontief model cannot be derived from (3): this operation is a deadlock.[7] In order to get out of this difficulty, prices should be considered. However, from (8), it comes

$$\mathbf{A} = \hat{\mathbf{p}}\bar{\mathbf{Z}}\left[\hat{\bar{\mathbf{x}}}\right]^{-1}\left[\hat{\mathbf{p}}\right]^{-1} = \hat{\mathbf{p}}\bar{\mathbf{A}}\left[\hat{\mathbf{p}}\right]^{-1}$$

Therefore, the product $\sum_{i=1}^{n} p_i a_{ij}$ for any j, i.e., $\mathbf{p}'\mathbf{A}$, turns out to be

$$\mathbf{p}'\hat{\mathbf{p}}\bar{\mathbf{A}}\hat{\mathbf{p}}^{-1} = \mathbf{s}'\left[\hat{\mathbf{p}}\right]^{2}\bar{\mathbf{A}}^{(t)}\hat{\mathbf{p}}^{-1}$$

input-output tables because it is impossible to aggregate commodities easily. However, we neglect this question because we want to discuss the theoretical foundations of the Leontief model, and not to conduct a discussion about national accounting.

[7] In other words, the column sums of $\mathbf{L}\left[\mathbf{I} - \mathbf{A}\right]^{-1}$ are identically equal: from the balance (10), it holds that $\mathbf{s}'\mathbf{L}\left[\mathbf{I} - \mathbf{A}\right] \equiv \mathbf{s}'$. This property means that it always needs, directly and indirectly, \$100 of labor for producing \$100 of any commodity.

where $\hat{\mathbf{p}}^2$, i.e., the diagonal matrix of terms p_i^2, is obviously an economic nonsense. We deduce that applying prices over technical coefficients is not allowed.

Fortunately, there is a classical solution, in dynamics. It has two steps:
(i) Two distinct time periods are considered, which is a very usual procedure in national accounting: the *base time period* (denoted 0) and the *current time period* (denoted T).
(ii) The variation of prices are calculated between time period T and time period 0, which conducts to introduce price indexes $\boldsymbol{\pi}$ (and $\boldsymbol{\pi}_v$ for labor) that are simply defined as the ratio of the prices of the base time period to those of the current time period. They are called also deflators. Now, we are able to derive the dual of the Leontief model.

The dual Leontief model We write equation (6) for the current period, that is,

$$\mathbf{p}^{(T)\prime}\bar{\mathbf{A}}^{(T)} + \mathbf{p}_v^{(T)\prime}\bar{\mathbf{L}}^{(T)} = \mathbf{p}^{(T)\prime} \tag{11}$$

As $\bar{\mathbf{A}}^{(T)}$ and $\bar{\mathbf{L}}^{(T)}$ are coefficients independent to the endogenous variable $\mathbf{p}^{(T)}$ and to the exogenous variable $\mathbf{p}_v^{(T)}$, this expression can be derived for $\mathbf{p}^{(T)}$ and $\mathbf{p}_v^{(T)}$, that is, we are allowed to calculate the variation of price between the current period and the base period (generally negative in time of inflation):

$$\Delta\mathbf{p}^{(T,0)\prime}\bar{\mathbf{A}}^{(T)} + \Delta\mathbf{p}_v^{(T,0)\prime}\bar{\mathbf{L}}^{(T)} = \Delta\mathbf{p}^{(T,0)\prime} \tag{12}$$

Subtracting (12) to (11) yields:

$$\tilde{\mathbf{p}}^{(0)\prime}\bar{\mathbf{A}}^{(T)} + \tilde{\mathbf{p}}_v^{(0)\prime}\bar{\mathbf{L}}^{(T)} = \tilde{\mathbf{p}}^{(0)\prime} \tag{13}$$

by denoting

$$\tilde{\mathbf{p}}^{(0)} = \mathbf{p}^{(T)} - \Delta\mathbf{p}^{(T,0)} \tag{14}$$

and

$$\tilde{\mathbf{p}}_v^{(0)} = \mathbf{p}_v^{(T)} - \triangle\mathbf{p}_v^{(T,0)} \tag{15}$$

We call $\tilde{\mathbf{p}}^{(0)}$ (and $\tilde{\mathbf{p}}_v^{(0)}$ for labor) the *Leontief prices*.
We can go to the index prices by rewriting equation (13) as

$$\tilde{\mathbf{p}}^{(0)\prime}\left[\hat{\mathbf{p}}^{(T)}\right]^{-1}\left(\hat{\mathbf{p}}^{(T)}\bar{\mathbf{A}}^{(T)}\left[\hat{\mathbf{p}}^{(T)}\right]^{-1}\right) + \tilde{\mathbf{p}}_v^{(0)\prime}\left[\hat{\mathbf{p}}_v^{(T)}\right]^{-1}\left(\hat{\mathbf{p}}_v^{(T)}\bar{\mathbf{L}}^{(T)}\left[\hat{\mathbf{p}}^{(T)}\right]^{-1}\right)$$

$$= \tilde{\mathbf{p}}^{(0)\prime} \left[\hat{\mathbf{p}}^{(T)}\right]^{-1}$$

which is equivalent to

$$\tilde{\boldsymbol{\pi}}' \mathbf{A}^{(T)} + \tilde{\boldsymbol{\pi}}_v' \mathbf{L}^{(T)} = \tilde{\boldsymbol{\pi}}' \tag{16}$$

where $\mathbf{L}^{(T)} = \hat{\mathbf{p}}_v^{(T)} \bar{\mathbf{L}}^{(T)} \left[\hat{\mathbf{p}}^{(T)}\right]^{-1}$ and

$$\hat{\tilde{\boldsymbol{\pi}}} = \hat{\tilde{\mathbf{p}}}^{(0)} \left[\hat{\mathbf{p}}^{(T)}\right]^{-1}$$

for commodities and

$$\hat{\tilde{\boldsymbol{\pi}}}_v = \hat{\tilde{\mathbf{p}}}_v^{(0)} \left[\hat{\mathbf{p}}_v^{(T)}\right]^{-1}$$

for labor.[8] Now, with price indexes, the product $\tilde{\boldsymbol{\pi}}' \mathbf{A}^{(T)}$ takes sense .[9] Equation (16) is the dual Leontief model.

Remark. Introducing the price indexes π on a model with monetary data of the current period—$\mathbf{Z}^{(T)}$—is mathematically the same thing than introducing the base prices $\mathbf{p}^{(0)}$ on a model with physical quantities of the current period—$\bar{\mathbf{Z}}^{(T)}$—because $\hat{\tilde{\boldsymbol{\pi}}} \mathbf{Z}^{(T)} = \hat{\tilde{\mathbf{p}}}^{(0)} \bar{\mathbf{Z}}^{(T)}$.

From (16), as $\mathbf{A}^{(T)}$ and $\mathbf{L}^{(T)}$ are assumed stable, one deduces that the Leontief price indexes $\tilde{\boldsymbol{\pi}}$ are formed from the Leontief labor price index, $\tilde{\boldsymbol{\pi}}_v$, but by using the monetary production structure and the monetary labor coefficients of the *current* period

$$\tilde{\boldsymbol{\pi}}' = \tilde{\boldsymbol{\pi}}_v' \mathbf{L}^{(T)} \left[\mathbf{I} - \mathbf{A}^{(T)}\right]^{-1} \tag{17}$$

From (13) it comes also

$$\tilde{\mathbf{p}}^{(0)\prime} = \tilde{\mathbf{p}}_v^{(0)\prime} \bar{\mathbf{L}}^{(T)} \left[\mathbf{I} - \bar{\mathbf{A}}^{(T)}\right]^{-1} \tag{18}$$

that is, the **base** Leontief prices $\tilde{\mathbf{p}}^{(0)}$ are derived from the **base** prices of labor $\tilde{\mathbf{p}}_v^{(0)}$ and from the physical coefficients of the **current** period, i.e., production structure $\bar{\mathbf{A}}^{(T)}$ and physical structure of labor costs $\bar{\mathbf{L}}^{(T)}$.

Remark. (i) It is obvious that, at the current time period ($t = T$), the Leontief price indexes are identically equal to 1: $\tilde{\boldsymbol{\pi}}' = \mathbf{s}' \mathbf{L}^{(T)} \left[\mathbf{I} - \mathbf{A}^{(T)}\right]^{-1} \equiv \mathbf{s}'$ because of the balance (10). (ii) Leontief does as if monetary coefficients were physical

[8] On more sophisticated price indexes, see (Fisher and Shell 1997).

[9] The model remains static, with two time periods 0 and T; it is absolutely not dynamic in the sense of the so-called dynamic Leontief model (1970), where the primal equation in monetary terms is $\mathbf{x}^{(t)} - \mathbf{A}^{(t)} \mathbf{x}^{(t)} - \mathbf{B}^{(t)} \left(\mathbf{x}^{(t+1)} - \mathbf{x}^{(t)}\right) = \mathbf{f}^{(t)}$, $\mathbf{B}^{(t)}$ being the matrix of capital coefficients.

coefficients, that is, $\mathbf{A}^{(T)} \equiv \bar{\mathbf{A}}^{(T)}$, which is rather abusive, and price indexes were prices, i.e., $\tilde{\boldsymbol{\pi}} = \tilde{\mathbf{p}}^{(0)}$ (and $\tilde{\boldsymbol{\pi}}_v = \tilde{\mathbf{p}}_v^{(0)}$ for labor), which amounts to pose a hypothesis on the prices of the current time period, as $\mathbf{p}^{(T)} = \mathbf{s}$.

3 Dual Leontief model vs. dual production-prices model

We may compare the dual of the Leontief model to the dual of a production-prices model based on a physical matrix for time period T. We rewrite (7), for the base year:

$$\mathbf{p}^{(0)\prime} = \mathbf{p}_v^{(0)\prime}\bar{\mathbf{L}}^{(0)} \left[\mathbf{I} - \bar{\mathbf{A}}^{(0)}\right]^{-1} \tag{19}$$

which defines the *true* (base) *prices*. They can be compared to the Leontief prices given by (18).

One remarks that equations (19) and (18) only differ by the fact that $\bar{\mathbf{A}}^{(0)}$ and $\bar{\mathbf{L}}^{(0)}$ replace $\bar{\mathbf{A}}^{(T)}$ and $\bar{\mathbf{L}}^{(T)}$ respectively.

Now, we have the tools to answer to the following question: is the Leontief model a production-prices model? That is: under what conditions the Leontief prices are equal to the true current prices? We assume that the vector of labor prices is the same in both models, i.e., $\mathbf{p}_v^{(0)} = \tilde{\mathbf{p}}_v^{(0)}$; the price of labor being exogenous in the context of the input-out model, this is a normal hypothesis: from (15), it simply means that we are able to forecast correctly the variation $\triangle\mathbf{p}_v^{(T,0)}$ of the price of labor from its value $\mathbf{p}_v^{(T)}$ at the current period.

We need to recall that the matrix $\bar{\mathbf{L}}^{(t)} \left[\mathbf{I} - \bar{\mathbf{A}}^{(t)}\right]^{-1}$ is called the interindustry matrix of direct and indirect quantities of labor (measured in physical units) incorporated per unit of physical output because it holds that

$$\left[\mathbf{I} - \bar{\mathbf{A}}^{(t)}\right]^{-1} = \left[\mathbf{I} + \bar{\mathbf{A}}^{(t)} + \bar{\mathbf{A}}^{(t)^2} + ... + \bar{\mathbf{A}}^{(t)^n} + ...\right]$$

then

$$\bar{\mathbf{L}}^{(t)} \left[\mathbf{I} - \bar{\mathbf{A}}^{(t)}\right]^{-1} = \bar{\mathbf{L}}^{(t)} + \bar{\mathbf{L}}^{(t)}\bar{\mathbf{A}}^{(t)} + \bar{\mathbf{L}}^{(t)}\bar{\mathbf{A}}^{(t)^2} + ... + \bar{\mathbf{L}}^{(t)}\bar{\mathbf{A}}^{(t)^n} + ...$$

Therefore, the term $\{i, j\}$ of the matrix $\bar{\mathbf{L}}^{(t)} \left[\mathbf{I} - \bar{\mathbf{A}}^{(t)}\right]^{-1}$ is the direct and indirect quantity of labor incorporated in each input i per unit of physical output produced by sector j.

Definition. The dual Leontief model is *coherent* if it can be confused to a dual production-prices model, that is, $\tilde{\mathbf{p}}^{(0)} = \mathbf{p}^{(0)}$.

Theorem. *The Leontief model is coherent if and only if the interindustry matrix of direct and indirect quantities of labor is stable over time, that is:*

$$\bar{\mathbf{L}}^{(T)} \left[\mathbf{I} - \bar{\mathbf{A}}^{(T)}\right]^{-1} = \bar{\mathbf{L}}^{(0)} \left[\mathbf{I} - \bar{\mathbf{A}}^{(0)}\right]^{-1} \tag{20}$$

Proof. $\tilde{\mathbf{p}}^{(0)} = \mathbf{p}^{(0)}$

$$\Leftrightarrow \tilde{\mathbf{p}}_v^{(0)\prime} \bar{\mathbf{L}}^{(T)} \left[\mathbf{I} - \bar{\mathbf{A}}^{(T)}\right]^{-1} = \mathbf{p}_v^{(0)\prime} \bar{\mathbf{L}}^{(0)} \left[\mathbf{I} - \bar{\mathbf{A}}^{(0)}\right]^{-1}$$

As labor is exogenous in the Leontief model, we can pose the assumption that: $\mathbf{p}_v^{(0)} = \tilde{\mathbf{p}}_v^{(0)}$.

$$\Leftrightarrow \mathbf{p}_v^{(0)\prime} \bar{\mathbf{L}}^{(T)} \left[\mathbf{I} - \bar{\mathbf{A}}^{(T)}\right]^{-1} = \mathbf{p}_v^{(0)\prime} \bar{\mathbf{L}}^{(0)} \left[\mathbf{I} - \bar{\mathbf{A}}^{(0)}\right]^{-1}$$

which must be true for any $\mathbf{p}_v^{(0)}$, that is, by equating for $\mathbf{p}_v^{(0)}$[10]

$$\bar{\mathbf{L}}^{(T)} \left[\mathbf{I} - \bar{\mathbf{A}}^{(T)}\right]^{-1} = \bar{\mathbf{L}}^{(0)} \left[\mathbf{I} - \bar{\mathbf{A}}^{(0)}\right]^{-1}$$

□

Definition. The term j of the row vector $\mathbf{s}'\bar{\mathbf{L}}^{(t)} \left[\mathbf{I} - \bar{\mathbf{A}}^{(t)}\right]^{-1}$ is the direct and indirect quantity of labor (measured in physical units) incorporated per unit of physical output produced in sector j for time period t.

This concept is called the *vertically integrated coefficients of labor*, which for Pasinetti (1977, chap. 5, subsection 2 of the appendix) corresponds to the *Marxian values*.

Corollary 1. *If the Leontief model is coherent, then the vertically integrated coefficients of labor are stable over time, that is,*

$$\mathbf{s}'\bar{\mathbf{L}}^{(T)} \left[\mathbf{I} - \bar{\mathbf{A}}^{(T)}\right]^{-1} = \mathbf{s}'\bar{\mathbf{L}}^{(0)} \left[\mathbf{I} - \bar{\mathbf{A}}^{(0)}\right]^{-1} \tag{21}$$

Proof. The proof is obvious by premultiplying (20) by $\mathbf{s}'$. □

[10] Two polynomial expression are equal if and only if their coefficients are equal.

The stability over time of the vertically integrated coefficients of labor means that any sector j needs as much labor for producing one physical unit of commodity in the current period as in the base period: no productivity gains for what concerns labor. The Leontief production function is linear but nothing would *a-priori* prevent the evolution of the coefficients for reflecting such productivity gains; the Corollary 1 excludes them. Remark that assuming or obtaining the stability of the matrix $\bar{\mathbf{L}}^{(t)}\left[\mathbf{I}-\bar{\mathbf{A}}^{(t)}\right]^{-1}$ is much stronger than assuming the stability of the vector $\mathbf{s}'\bar{\mathbf{L}}^{(t)}\left[\mathbf{I}-\bar{\mathbf{A}}^{(t)}\right]^{-1}$. Indeed, the stability of $\bar{\mathbf{L}}^{(t)}\left[\mathbf{I}-\bar{\mathbf{A}}^{(t)}\right]^{-1}$ implies those of $\mathbf{s}'\bar{\mathbf{L}}^{(t)}\left[\mathbf{I}-\bar{\mathbf{A}}^{(t)}\right]^{-1}$ but the reciprocal proposition is false.

The cases where equation (20) holds can be qualified as *pseudo physical stability*. However, the true physical stability, those of $\bar{\mathbf{A}}$ and $\bar{\mathbf{L}}$, implies the pseudo physical stability, as shown by the following corollary:

Corollary 2. *If two of the three following properties hold, the third one holds also:*

$$\begin{cases} \bar{\mathbf{L}}^{(T)}\left[\mathbf{I}-\bar{\mathbf{A}}^{(T)}\right]^{-1} = \bar{\mathbf{L}}^{(0)}\left[\mathbf{I}-\bar{\mathbf{A}}^{(0)}\right]^{-1} \\ \bar{\mathbf{A}}^{(T)} = \bar{\mathbf{A}}^{(0)} \\ \bar{\mathbf{L}}^{(T)} = \bar{\mathbf{L}}^{(0)} \end{cases}$$

Proof. If $\bar{\mathbf{L}}^{(T)} = \bar{\mathbf{L}}^{(0)}$ and $\bar{\mathbf{A}}^{(T)} = \bar{\mathbf{A}}^{(0)}$ hold simultaneously, then (20) holds. If $\bar{\mathbf{L}}^{(T)} = \bar{\mathbf{L}}^{(0)}$ and (20) hold simultaneously, then

$$\left[\mathbf{I}-\bar{\mathbf{A}}^{(T)}\right]^{-1} = \left[\mathbf{I}-\bar{\mathbf{A}}^{(0)}\right]^{-1} \Leftrightarrow \bar{\mathbf{A}}^{(T)} = \bar{\mathbf{A}}^{(0)}$$

If $\bar{\mathbf{A}}^{(T)} = \bar{\mathbf{A}}^{(0)}$ and (20) hold simultaneously, then $\bar{\mathbf{L}}^{(T)} = \bar{\mathbf{L}}^{(0)}$. □

Corollary 3. $\tilde{\mathbf{p}}^{(0)} = \mathbf{p}^{(0)}$ *holds if all physical coefficients* $\bar{\mathbf{L}}$ *and* $\bar{\mathbf{A}}$ *are stable*

$$\tilde{\mathbf{p}}^{(0)} = \mathbf{p}^{(0)} \Leftarrow \begin{cases} \bar{\mathbf{A}}^{(T)} = \bar{\mathbf{A}}^{(0)} \\ \bar{\mathbf{L}}^{(T)} = \bar{\mathbf{L}}^{(0)} \end{cases}$$

Proof. The proof directly follows from the corollary 2. □

Assuming that the physical production coefficients are stable over time, i.e., $\bar{\mathbf{A}}^{(T)} = \bar{\mathbf{A}}^{(0)}$, is a very strong and rather unrealistic hypothesis. In this case, the eventual study of the structural change becomes completely uninteresting: the physical structural change becomes equal to zero ($\bar{\mathbf{A}}^{(T)} = \bar{\mathbf{A}}^{(0)}$ by hypothesis) and the structural change in currency units only turns out to be a price effect.[11]

Remark. In equations 11, 12 and 13, one could revert the role of the base period and the current period, which gives:

$$\tilde{\mathbf{p}}^{(T)\prime}\bar{\mathbf{A}}^{(0)} + \tilde{\mathbf{p}}_v^{(T)\prime}\bar{\mathbf{L}}^{(0)} = \tilde{\mathbf{p}}^{(T)\prime} \tag{22}$$

by denoting

$$\tilde{\mathbf{p}}^{(T)} = \mathbf{p}^{(0)} + \triangle\mathbf{p}^{(0)} \tag{23}$$

and

$$\tilde{\mathbf{p}}_v^{(T)} = \mathbf{p}_v^{(0)} + \triangle\mathbf{p}_v^{(0)} \tag{24}$$

and,

$$\tilde{\boldsymbol{\sigma}}'\mathbf{A}^{(T)} + \tilde{\boldsymbol{\sigma}}_v'\mathbf{L}^{(T)} = \tilde{\boldsymbol{\sigma}}' \tag{25}$$

by denoting

$$\hat{\boldsymbol{\sigma}}' = \hat{\mathbf{p}}^{(T)\prime}\left[\hat{\mathbf{p}}^{(0)}\right]^{-1}$$

for commodities and

$$\hat{\boldsymbol{\sigma}}_v' = \hat{\mathbf{p}}_v^{(T)\prime}\left[\hat{\mathbf{p}}_v^{(0)}\right]^{-1}$$

for labor. From (22) it comes also $\tilde{\mathbf{p}}^{(T)\prime} = \tilde{\mathbf{p}}_v^{(T)\prime}\bar{\mathbf{L}}^{(0)}\left[\mathbf{I} - \bar{\mathbf{A}}^{(0)}\right]^{-1}$, that is, the **current** Leontief prices $\tilde{\mathbf{p}}^{(T)}$ are derived from the **current** prices of labor $\tilde{\mathbf{p}}_v^{(0)}$ but also from the physical coefficients of the **base** time period, i.e., production structure $\bar{\mathbf{A}}^{(0)}$ and physical structure of labor costs $\bar{\mathbf{L}}^{(0)}$. This way of doing is equivalent but far from national accounting practice. But that does not change the Theorem and the Corollaries 1, 2 and 3.

4 Conclusion

The Leontief model is generally considered as belonging to the category of the "production-prices models", aside of the Ricardian, Marxian and Sraffaian models. Nevertheless, this paper clearly demonstrated that this is a superficial point of view. The Leontief model actually uses two periods as it should handle price indexes: the Leontief price indexes—the deflators—solve the model with the coefficients of the current time period; the production-prices model that corresponds to the Leontief model is monoperiodic: the current prices solve the

[11]This analysis could be extended to the case of multiple factors even if the Leontief model typically considers only one factor, labor.

production-prices model with the coefficients of the base time period.

We show that both models diverge generally unless a very strong assumption is done: the interindustry matrix of direct and indirect quantities of labor incorporated per unit of physical output is stable over time. This implies that the vertically integrated coefficients of labor are stable. This assumption is satisfied when the production coefficients in physical terms and the physical coefficients of labor are stable over time, two very strong assumptions. The results are unchanged if the role of the base period and the current period are reverted.

We conclude that the Leontief model is **generally not** a production-prices model, unless some quite Classical-Marxian assumptions are done.

5 References

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